$\langle V, \vec{A}\rangle$. (Here, Karp refined the construction so as to show that $F_s^{\vec{A}} \upharpoonright M = F_s^M$ whenever $M = \langle |M|, \in, A_1 \cap |M|, \ldots, A_n \cap |M|\rangle$ is transitive and closed under function primitive recursive in $A_1, \ldots, A_n$. It can also be shown that $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ is closed under functions primitive recursive in $A_1, \ldots, A_n$ iff $|M|$ is primitive recursive closed and M is amenable, (i.e. $x \cap A_i \in M$ for all $x \in M$, $v = 1, \ldots, n$).

A full account of these results can be found in [PR] or [AS].

We can now state the uniformity involved in Lemma 2.2.19: Let $A_i \subset V$ be primitive recursive in $B_1, \ldots, B_m$ with primitive recursive def s_i in $\dot{b}_1, \ldots, \dot{b}_m$ $(i = 1, \ldots, m)$. Let f be primitive recursive in $A_1, \ldots, A_n$ with primitive recursive definition s in $\dot{a}_1, \ldots, \dot{a}_n$. Then f is primitive recursive in $B_1, \ldots, B_n$ by a primitive recursive definition s' in $\dot{b}_1, \ldots, \dot{b}_m$. s' is *uniform* in the sense that it depends only on $s_1, \ldots, s_n$ and s, not on $B_1, \ldots, B_m$. In fact, the induction on the schemata in s implicitly describes an algorithm for a function

$$s_1, \ldots, s_m, s \mapsto s'$$

with the following property: Let $B_1, \ldots, B_m$ be any classes. Let s_i define g_i from $\vec{B}(i = 1, \ldots, n)$. Set: $A_i = \{x | g_i(x) \neq 0\}$ in $i = 1, \ldots, n$. Let f be the function defined by s from $\vec{A}$. Then s' defines f from $\vec{B}$.

Note $\langle H_\omega, \in\rangle$ is an admissible structure; hence $F_s \upharpoonright H_\omega = f_s^{H_\omega}$. This shows that the constant function ω is not primitive recursive, since $\omega \notin H_\omega$. It can be shown that $f : \omega \to \omega$ is primitive recursive in the sense of ordinary recursion theory iff

$$f^*(x) = \begin{cases} f(x) \text{ if } x \in \omega \\ 0 \text{ if not} \end{cases}$$

is primitive recursive over H_ω. Conversely, there is a primitive recursive map $\sigma : H_\omega \leftrightarrow \omega$ such that $f : H_\omega \to H_\omega$ is primitive recursive over H_ω iff $\sigma f \sigma^{-1}$ is primitive recursive in sense of ordinary recursion theory.

1.3 Ill founded ZF^- models

We now prove a lemma about arbitrary (possibly ill founded) models of ZF^- (where the language of ZF^- may contain predicates other than $\in$). Let $\mathbb{A} = \langle A, \in_{\mathbb{A}}, B_1, \ldots, B_n\rangle$ be such a model. For $X \subset A$ we of course write $\mathbb{A}|X = \langle X, \in_A \cap X^2, \ldots\rangle$. By the *well founded core* of $\mathbb{A}$ we mean the set of all $v \in \mathbb{A}$ such that $\in_{\mathbb{A}} \cap C(x)^2$ is well founded, where $C(x)$ is the closure of $\{x\}$ under $\in_{\mathbb{A}}$. Let wfc($\mathbb{A}$) be the restriction $\mathbb{A}|C$ of $\mathbb{A}$ to its well founded core C. Then wfc($\mathbb{A}$) is a well founded structure satisfying the axiom of extensionality, and is, therefore, isomorphic to a transitive structure. Hence $\mathbb{A}$ is isomorphic to a structure $\mathbb{A}'$ such that wfc($\mathbb{A}'$) is transitive (i.e. $\text{wfc}(\mathbb{A}') = \langle A', \in, m\rangle$ where A' is transitive). We call such $\mathbb{A}'$ *grounded*, defining:

Definition 1.3.1. $\mathbb{A} = \langle A, \in_{\mathbb{A}}, \ldots\rangle$ is *grounded* iff wfc($\mathbb{A}$) is transitive.

Note. Elsewhere we have called these models "solid" instead of "grounded". We avoid that usage here, however, since *solidity* — in quite another sense — is an important concept in inner model theory.

By the argument just given, every consistent set of sentences in ZF^- has a grounded model. Clearly

(1) $\omega \subset \text{wfc}(\mathbb{A})$ if $\mathbb{A}$ is grounded.

For any ZF^- model $\mathbb{A}$ we have:

(2) If $x \in \mathbb{A}$ and $\{z | z \in_{\mathbb{A}} x\} \subset \text{wfc}(\mathbb{A})$, then $x \in \text{wfc}(\mathbb{A})$.

Proof: $C(x) = \{x\} \cup \bigcup\{C(z) | z \in_{\mathbb{A}} x\}$. QED

By Σ_0–absoluteness we have:

(3) Let $\mathbb{A}$ be grounded. Let φ be Σ_0 and let $x_1, \ldots, x_n \in \text{wfc}(\mathbb{A})$. Then

$$\text{wfc}(\mathbb{A}) \models \varphi[\vec{x}] \leftrightarrow \mathbb{A} \models \varphi[\vec{x}].$$

By $\in$–induction on $x \in \text{wfc}(\mathbb{A})$ it follows that the rank function is absolute:

(4) $\text{rn}(x) = \text{rn}^{\mathbb{A}}(x)$ for $x \in \text{wfc}(\mathbb{A})$ if $\mathbb{A}$ is grounded.

The converse also holds:

(5) Let $\text{rn}^{\mathbb{A}}(x) \in \text{wfc}(\mathbb{A})$. Then $x \in \text{wfc}(\mathbb{A})$.

Proof: Let $r = \text{rn}^{\mathbb{A}}(x)$. Then r is an ordinal by (3). Assume that r is the least counterexample. Then $\text{rn}^{\mathbb{A}}(z) < r$ for $z \in_{\mathbb{A}} x$. Hence $\{z | z \in_{\mathbb{A}} x\} \subset \text{wfc}(\mathbb{A})$ and $x \in \text{wfc}(\mathbb{A})$ by (2).

Contradiction! QED

We now prove:

Lemma 1.3.1. *Let* $\mathbb{A}$ *be grounded. Then* $\mathrm{wfc}(\mathbb{A})$ *is admissible.*

Proof: Axiom (1) and axiom (2) (Σ_0–subsets) follow trivially from (3). We verify the axiom of Σ_0 collection. Let $R(x,y)$ be $\underline{\Sigma}_0(\mathrm{wfc}(\mathbb{A}))$. Let $u \in \mathrm{wfc}(\mathbb{A})$ such that $\bigwedge x \in u \bigvee y R(x,y)$. It suffices to show:

Claim: $\bigvee v \bigwedge x \in u \bigvee y \in v R(x,y)$.

Let R' be $\underline{\Sigma}_0(\mathbb{A})$ by the same definition in the same parameters as R. Then $R = R' \cap \mathrm{wfc}(\mathbb{A})^2$ by (3). If $\mathbb{A} = \mathrm{wfc}(\mathbb{A})$, there is nothing to prove, so suppose not. Then there is $r \in \mathrm{On}^{\mathbb{A}}$ such that $r \notin \mathrm{wfc}(\mathbb{A})$. Hence

$$\mathbb{A} \models rn(y) < r \text{ for all } y \in \mathrm{wfc}(\mathbb{A})$$

by (4). Hence there is an $r \in \mathrm{On}^{\mathbb{A}}$ such that

(6) $\bigwedge x \in u \bigvee y (R'(x,y) \wedge \mathbb{A} \models rn(y) < r)$

Since $\mathbb{A}$ models ZF^-, there must be a least such r. But then:

(7) $r \in \mathrm{wfc}(\mathbb{A})$.

Since by (2) there would otherwise be an r' such that $\mathbb{A} \models r' < r$ and $r' \notin \mathrm{wfc}(\mathbb{A})$. Hence (6) holds for r', contradicting the minimality of r.

QED (7)

But there is w such that

(8) $\bigwedge x \in u \bigvee y \in w (R'(x,y) \wedge rn(y) < r)$.

Let $\mathbb{A} \models v = \{y \in w | rn(y) < r\}$. Then $rn^{\mathbb{A}}(v) \leq r$. Hence $rn^{\mathbb{A}}(v) \in \mathrm{wfc}(\mathbb{A})$ and $v \in \mathrm{wfc}(\mathbb{A})$ by (5). But:

$$\bigwedge x \in u \bigvee y \in v Rxy.$$

QED (Lemma 1.3.1)

As immediate corollaries we have:

Corollary 1.3.2. *Let* $\delta = \mathrm{On} \cap \mathrm{wfc}(\mathbb{A})$*. Then* $L_\delta(u)$ *is admissible whenever* $u \in \mathrm{wfc}(\mathbb{A})$.

Corollary 1.3.3. $L_\delta^A = \langle L_\delta[A], A \cap L_\delta[A] \rangle$ *is admissible whenever* $A \in \underline{\Sigma}_\omega(\mathbb{A})$ *(since* $\langle \mathbb{A}, A \rangle$ *is a* ZF^- *model.*

Note. It is clear from the proof of lemma 1.3.1 that we can replace ZF^- by KP (Kripke–Platek set theory). In this form Lemma 1.3.1 is known as *Ville's Lemma.*