Proof: X satisfies the extensionality axiom. Hence by Mostowski's isomorphism theorem there is $\pi : \overline{U} \overset{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive. Now let f be rud and $x_1, \ldots, x_n \in \overline{U}$. Then there is $y' \in X$ such that $y' = f(\pi(\vec{x}))$, since $X \prec_{\Sigma_1} U$. Let $\pi(y) = y'$. Then $y = f(\vec{x})$, since the condition '$y = f(\vec{x})$' is Σ_0 and π is Σ_1–preserving. QED (Lemma 2.2.19)

The condensation lemma for rud closed $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ is much weaker, however. We state it for the case $n = 1$.

Lemma 2.2.20. *Let* $M = \langle |M|, \in, A \rangle$ *be transitive and* rud *closed. Let* $X \prec_{\Sigma_1} M$. *There is an isomorphism* $\pi : \overline{M} \overset{\sim}{\longleftrightarrow} X$, *where* $\overline{M} = \langle |\overline{M}|, \in, \overline{A} \rangle$ *is transitive and* rud *closed. Moreover:*

(a) $\pi(\overline{A} \cap x) = A \cap \pi(x)$

(b) Let f *be* rud *in* A. *Let* f *be characterized by:* $f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x}))$, *where* f_0, f_1 *are* rud. *Set:* $\overline{f}(\vec{x}) =: f_0(\vec{x}, \overline{A} \cap f_1(\vec{x}))$. *Then:*

$$\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x})).$$

The proof is left to the reader.

2.3 The J_α hierarchy

We are now ready to introduce the alternative to Gödel's constructible hierarchy which we had promised in §1. We index it by ordinals from the class Lm of limit ordinals.

Definition 2.3.1.

$$\begin{aligned} J_\omega &= \mathrm{Rud}(\emptyset) \\ J_{\beta+\omega} &= \mathrm{Rud}(J_\beta) \text{ for } \beta \in \mathrm{Lm} \\ J_\lambda &= \bigcup_{\gamma<\lambda} J_\gamma \text{ for } \lambda \text{ a limit point of } \mathrm{Lm} \end{aligned}$$

It can be shown that $L = \bigcup\limits_{\alpha} J_\alpha$ and, indeed, that $L_\alpha = J_\alpha$ for a great many α (for instance closed α). Note that $J_\omega = L_\omega = H_\omega$.

By §2 Corollary 2.2.14 we have:

$$\mathbb{P}(J_\alpha) \cap J_{\alpha+\omega} = \mathrm{Def}(J_\alpha),$$

which pinpoints the resemblance of the two hierarchies. However, we shall not dwell further on the relationship of the two hierarchies, since we intend to consequently employ the J–hierarchy in the rest of this book. As usual, we shall often abuse notation by not distinguishing between J_α and $\langle J_\alpha, \in \rangle$.

Lemma 2.3.1. $\mathrm{rn}(J_\alpha) = \mathrm{On} \cap J_\alpha = \alpha$.

Proof: By induction on $\alpha \in \mathrm{Lm}$. For $\alpha = \omega$ it is trivial. Now let $\alpha = \beta + \omega$, where $\beta \in \mathrm{Lm}$. Then $\beta = \mathrm{On} \cap J_\beta \in \mathrm{Def}(J_\beta) \subset J_\alpha$. Hence $\beta + n \in J_\alpha$ for $n < \omega$ by rud closure. But $\mathrm{rn}(J_\alpha) \leq \beta + \omega = \alpha$ since J_α is the rud closure of $J_\alpha \cup \{J_\alpha\}$. Hence $\mathrm{On} \cap J_\alpha = \alpha = \mathrm{rn}(J_\alpha)$.

If α is a limit point of Lm the conclusion is trivial. QED (Lemma 2.3.1)

To make our notation simpler, define

Definition 2.3.2. $\mathrm{Lm}^* =$ the limit points of Lm.

It is sometimes useful to break the passage from J_α to $J_{\alpha+\omega}$ into ω many steps. Any way of doing this will be rather arbitrary, but we can at least do it in a uniform way. As a preliminary, we use the basis theorem (§2 Theorem 2.2.15) to prove:

Lemma 2.3.2. *There is a* rud *function* $s : V \to V$ *such that for all* U*:*

(a) $U \subset s(U)$

(b) $\mathrm{rud}(U) = \bigcup_{n<\omega} s^n(U)$

(c) If U *is transitive, so is* $s(U)$.

Proof: Define rud functions $G_i (i = 0, 1, 2, 3)$ by:

$$\begin{aligned} G_0(x, y, z) &= (x, y) \\ G_1(x, y, z) &= (x, y, z) \\ G_2(x, y, z) &= \{x, (y, z)\} \\ G_3(x, y, z) &= x^* y \end{aligned}$$

Set:

$$s(U) =: U \cup \bigcup_{i=0}^{9} F_i^U U^2 \cup \bigcup_{i=0}^{3} G_i^U U^3.$$

(a) is then immediate, (b) is immediate by the basis theorem. We prove (c).

Let $a \in s(U)$. We claim: $a \subset s(U)$. There are 14 cases: $a \in U$, $a = F_i(x, y)$ for an $i = 0, \ldots, 8$, where $x, y \in U$, and $a = G_i(x, y, z)$ where $x, y, z \in U$ and $i = 0, \ldots, 3$. Each of the cases is quite straightforward. We give some example cases:

- $a = F(x, y) = x \otimes y$. If $z \in a$, then $z = (x', y')$ where $x' \in x,\ y' \in y$. But then $x', y' \in U$ by transitivity and $z = G_0(x', y', x') \in s(U)$.
- $a = F_3(x, y) = \{(w, z, v) | z \in x \wedge (u, v) \in y\}$. If $a' = (w, z, v) \in a$, then $w, z, v \in U$ by transitivity and $a' = G_1(w, z, v) \in s(U)$.
- $a = F_8(x, y)$. If $a' \in a$, then $a' = x^* z$ where $z \in y$. Hence $z \in U$ by transitivity and $a' = G_3(x, z, z) \in s(U)$.
- $a = G_0(x, y, z) = \{\{x\}, \{x, y\}\}$. Then $a \subset F_0'' U^2 \subset s(U)$.
- $a = G_1(x, y, z) = (x, y, z) = \{\{x\}, \{x, (y, z)\}\}$. Then $\{x\} = F_0(x, x) \in s(U)$ and $\{x, (y, z)\} = G_2(x, y, z) \in s(U)$. QED (Lemma 2.3.2)

If we then set:

Definition 2.3.3. $S(U) = s(U \cup \{U\})$ we get:

Corollary 2.3.3. *S is a* rud *function such that*

(a) $U \cup \{U\} \subset S(U)$

(b) $\bigcup\limits_{n<\omega} S^n(U) = \mathrm{Rud}(U)$

(c) If U is transitive, so is $S(U)$.

We can then define:

Definition 2.3.4.

$$\begin{aligned} &S_0 = \emptyset \\ &S_{\nu+1} = S(S_\nu) \\ &S_\lambda = \bigcup_{\nu<\lambda} S_\nu \text{ for limit } \lambda. \end{aligned}$$

Obviously then: $J_\gamma = S_\gamma$ for $\gamma \in \mathrm{Lm}$. (It would be tempting to simply define $J_\nu = S_\nu$ for all $\nu \in \mathrm{On}$. We avoid this, however, since it could lead to confusion: At successors ν the models S_ν do not have very nice properties. Hence we retain the convention that whenever we write J_α we mean α to be a limit ordinal.)

Each J_α has Σ_1 knowledge of its own genesis:

Lemma 2.3.4. *$\langle S_\nu | \nu < \alpha \rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof: $y = S_\nu \leftrightarrow \bigvee f(\varphi(f) \wedge y = f(\nu))$, where $\varphi(f)$ is the Σ_0 formula:

$$\begin{aligned}&f \text{ is a function} \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge f(0) = \emptyset\\ &\wedge \bigwedge \xi \in \mathrm{dom}(f)(\xi + 1 \in \mathrm{dom}(f) \to f(\xi+1) = S(f(\xi)))\\ &\wedge \bigwedge \lambda \in \mathrm{dom}(f|(\lambda \text{ is a limit} \to f(\lambda) = \bigcup f''\lambda).\end{aligned}$$

Thus it suffices to show that the existence quantifier can be restricted to J_α — i.e.

Claim $\langle S_\nu | \nu < \tau\rangle \in J_\alpha$ for $\tau < \alpha$.

Case 1 $\alpha = \omega$ is trivial.

Case 2 $\alpha = \beta + \omega,\ \beta \in \mathrm{Lm}$.
Then $\langle S_\nu | \nu < \beta\rangle \in \mathrm{Def}(J_\beta) \subset J_\alpha$. Hence $S_\beta = \bigcup_{\nu<\beta} S_\nu \in J_\alpha$. By rud closure it follows that $S_{\beta+n} \in J_\alpha$ for $n \subset w$. Hence $S \restriction \nu \in J_\alpha$ for $\nu < \alpha$. QED (Case 2)

Case 3 $\alpha \in \mathrm{Lm}^*$.
This case is trivial since if $\nu < \beta \in \alpha \cap \mathrm{Lm}$. Then $S \restriction \nu \in J_\beta \subset J_\alpha$. QED (Lemma 2.3.4)

We now use our methods to show that each J_α has a uniformly $\Sigma_1(J_\alpha)$ well ordering. We first prove:

Lemma 2.3.5. *There is a* rud *function $w : V \to V$ such that whenever r is a well ordering of u, then $w(u,r)$ is a well ordering of $s(u)$ which end extends r.*

Proof: Let r_2 be the r–lexicographic ordering of u^2:

$$\langle x, y\rangle r_2 \langle z, w\rangle \leftrightarrow (xrz \vee (x = z \wedge yrw)).$$

Let r_3 be the r–lexicographic ordering of u^3. Set:

$$u_0 = u,\ u_{1+i} = F_i'' u^2 \text{ for } i = 0, \ldots, 8,\ u_{10+i} = G_i'' u^3 \text{ for } i = 0, \ldots, 3.$$

Define a well ordering w_i of u_i as follows: $w_0 = r$, For $i = 0, \ldots, 9$ set

$$\begin{aligned}&xw_{1+i}y \leftrightarrow \bigvee a, b \in u^2(x = F_i(a) \wedge y = F_i(b) \wedge\\ &\wedge ar_2b \wedge \bigwedge a' \in u^2(a'r_2a \to x \neq F_i(a')) \wedge\\ &\wedge \bigwedge b' \in u^2(b'r_2b \to y \neq F_i(b')))\end{aligned}$$

For $i = 0, \ldots, 3$ let w_{10+i} have the same definitions with G_i in place of F_i and u^3, r_3 in place of u^2, r_2.

We then set:

$$w = w(u) = \{\langle x,y\rangle \in s(u)^2 | \bigvee_{i=0}^{13} ((xw_i y \wedge x,y \notin \bigcup_{h<i} u_n) \vee$$
$$\vee (x \in \bigcup_{h<i} u_n \wedge y \notin \bigcup_{n<i} u_n))\}$$

(where $\bigcup_{h<0} u_n = \emptyset$). QED (Lemma 2.3.5)

If r is a well ordering of u, then

$$r_u = \{\langle x,y\rangle | \langle x,y\rangle \in r \vee (x \in u \wedge y = u)\}$$

is a well ordering of $u \cup \{u\}$ which end extends r. Hence if we set:

Definition 2.3.5. $W(u,r) =: w(u \cup \{u\}, r_u)$.

We have:

Corollary 2.3.6. *W is a* rud *function such that whenever r is a well ordering of u, then $W(u,r)$ is a well ordering of $S(u)$ which end extends r.*

If we then set:

Definition 2.3.6.
$$<_{S_0} = \emptyset$$
$$<_{S_{\nu+1}} = W(S_\nu, <_{S_\nu})$$
$$<_{S_\lambda} = \bigcup_{\nu<\lambda} <_{S_\nu} \text{ for limit } \lambda,$$

it follows that $<_{S_\alpha}$ is a well ordering of S_α which end extends $<_{S_\nu}$ for all $\nu < \alpha$.

Definition 2.3.7. $<_\alpha = <_{J_\alpha} =: <_{S_\alpha}$ for $\alpha \in \mathrm{Lm}$.

Then $<_\alpha$ is a well ordering of J_α for $\alpha \in \mathrm{Lm}$.

By a close imitation of the proof of Lemma 2.3.4 we get:

Lemma 2.3.7. *$\langle <_{S_\nu} | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof:
$$y = <_{S_\nu} \leftrightarrow \bigvee f \bigvee g (\varphi(f) \wedge \psi(f,g) \wedge y = g(\nu))$$

where φ is as in the proof of Lemma 2.3.4 and ψ is the Σ_0 formula:

$$\begin{aligned}&g \text{ is a function} \wedge \mathrm{dom}(g) = \mathrm{dom}(f)\\ &\wedge g(0 = \emptyset \wedge \bigwedge \xi \in \mathrm{dom}(g) | \xi + 1 \in \mathrm{dom}(g) \to\\ &\to g(\xi+1) = W(f(\xi), g(\xi)))\\ &\wedge \bigwedge \lambda \in \mathrm{dom}(g)\ (\lambda \text{ is a limit} \to g(\lambda) = \bigcup g''\lambda).\end{aligned}$$

Just as before, we show that the existence quantifiers can be restricted to J_α. QED (Lemma 2.3.7)

But then:

Corollary 2.3.8. *$<_\alpha = \bigcup\limits_{\nu<\alpha} <_{S_\nu}$ is a well ordering of J_α which is uniformly $\Sigma_1(J_\alpha)$. Moreover $<_\alpha$ end extends $<_\nu$ for $\nu \in \mathrm{Lm}$, $\nu < \alpha$.*

Corollary 2.3.9. *u_α is uniformly $\Sigma_1(J_\alpha)$, where $u_\alpha(x) \simeq \{z | z <_\alpha x\}$.*

Proof:

$$y = u_\alpha(x) \leftrightarrow \bigvee \nu (x \in S_\nu \wedge y = \{z \in S_\nu | z <_{S_\nu} x\})$$

QED (Corollary 2.3.9)

Note. We shall often write $<_{J_\alpha}$ for $<_\alpha$. We also write $<_\infty$ or $<_J$ or $<_L$ for $\bigcup\limits_{\alpha\in\mathrm{On}} <_\alpha$. Then $<_L$ well orders L and is an end extension of $<_\alpha$.

We obtain a particularly strong form of Gödel's condensation lemma:

Lemma 2.3.10. *Let $X \prec_{\Sigma_1} J_\alpha$. Then there are $\overline{\alpha}, \pi$ such that $\pi : J_{\overline{\alpha}} \overset{\sim}{\longleftrightarrow} X$.*

Proof: By §2 Lemma 2.2.19 there is rud closed U such that U is transitive and $\pi : \overset{\sim}{\longleftrightarrow} X$. Note that the condition

$$S(f,\nu) \leftrightarrow: f = \langle S_\xi | \nu < \xi\rangle$$

is Σ_0, since:

$$\begin{aligned}S(f,\nu) \leftrightarrow (&f \text{ is a function} \wedge\\ &\wedge \mathrm{dom}(f) = \nu \wedge f(0) = \emptyset \text{ if } 0 < \nu \wedge\\ &\bigwedge \xi \in \mathrm{dom}(f)(\xi + 1 \in \mathrm{dom}(f) \to\\ &\to f(\xi+1) = S(f(\xi)))).\end{aligned}$$

Let $\overline{\alpha} = \mathrm{On} \cap U$ and let $\overline{\nu} < \overline{\alpha}$. Let $\pi(\overline{\nu}) = \nu$. Then $f = \langle S_\xi | \xi < \nu\rangle \in X$ since $X \prec_{\Sigma_1} J_\alpha$. Let $\pi(\overline{f}) = f$. Then $\overline{f} = \langle S_\xi | \xi < \overline{\nu}\rangle$, since $S(\overline{f}, \overline{\nu})$. But then $J_{\overline{\alpha}} = \bigcup\limits_{\xi<\overline{\alpha}} S_\xi \subset U$. But since π is Σ_1 preserving we know that

$$\begin{aligned}x \in U \to &\bigvee f, \nu \in U (S(f,\nu) \wedge x \in U f''\nu)\\ &\to x \in J_{\overline{\alpha}}.\end{aligned}$$

QED (Lemma 2.3.10)

Corollary 2.3.11. *Let* $\pi : J_{\overline{\alpha}} : J_{\overline{\alpha}} \to_{\Sigma_1} J_\alpha$. *Then:*

(a) $\nu < \tau \leftrightarrow \pi(\nu) < \pi(\tau)$ *for* $\nu, \tau < \overline{\alpha}$.

(b) $x <_L y \leftrightarrow \pi(x) <_L \pi(y)$ *for* $x, y \in J_{\overline{\alpha}}$.
Hence:

(c) $\nu \leq \pi(\nu)$ *for* $\nu < \overline{\alpha}$.

(d) $x \leq_L \pi(x)$ *for* $x \in J_{\overline{\alpha}}$.

Proof: (a), (b) follow by the fact that $< \cap J_\alpha^2$ and $<_L \cap J_\alpha^2 =<_\alpha$ are uniformly $\Sigma_1(J_\alpha)$. But if $\pi(\nu) < \nu$, then $\nu, \pi(\nu), \pi^2(\nu), \ldots$ would form an infinite decreasing sequence by (a). Hence (c) holds. Similarly for (d). QED (Corollary 2.3.11)

2.3.1 The J_α^A–hierarchy

Given classes $A_1, \ldots, A_n$ one can generalize the previous construction by forming the *constructible hierarchy* $\langle J_\alpha^{A_1,\ldots,A_n} | \alpha \in \mathrm{Lim}\rangle$ *relativized to* $A_1, \ldots, A_n$. We have this far dealt only with the case $n = 0$. We now develop the case $n = 1$, since the generalization to $n > 1$ is then entirely straightforward. (Moreover the case $n = 1$ is sufficient for most applications.)

Definition 2.3.8. Let $A \subset V$. $\langle J_\alpha^A | \alpha \in \mathrm{Lm}\rangle$ is defined by:

$$J_\alpha^A = \langle J_\alpha[A], \in, A \cap J_\alpha[A]\rangle$$
$$J_\omega[A] = \mathrm{Rud}_A(\emptyset) = H_\omega$$
$$J_{\beta+\omega}[A] = \mathrm{Rud}_A(J_\beta) \text{ for } \beta \in \mathrm{Lm}$$
$$J_\lambda[A] = \bigcup_{\nu<\lambda} J_\nu[A] \text{ for } \lambda \in \mathrm{Lm}^*$$

Note. $A \cap J_\alpha[A]$ is treated as an unary predicate.

Thus every J_α^A is rud closed. We set

Definition 2.3.9.
$$L[A] = J[A] = \bigcup_{\alpha\in\mathrm{On}} J_\alpha[A];$$
$$L^A = J^A = \langle L[A], \in, A \cap L[A]\rangle.$$

Note. that $J_\alpha[\emptyset] = J_\alpha$ for all $\alpha \in \mathrm{Lm}$.

Repeating the proof of Lemma 1.1.1 we get:

Lemma 2.3.12. $\mathrm{rn}(J^A_\alpha) = \mathrm{On} \cap J^A_\alpha = \alpha$.

We wish to break $J^A_{\alpha+\omega}$ into ω smaller steps, as we did with $J_{\alpha+\omega}$. To this end we define:

Definition 2.3.10. $S^A(u) = S(u) \cup \{A \cap u\}$.

Corresponding to Corollary 2.3.3 we get:

Lemma 2.3.13. *S^A is a function* rud *in A such that whenever u is transitive, then:*

(a) $u \cup \{u\} \cup \{A \cap u\} \subset S(u)$

(b) $\bigcup\limits_{n<\omega} (S^A)^n(u) = \mathrm{Rud}_A(u)$

(c) $S(u)$ is transitive.

Proof: (a) is immediate. (c) holds, since $S(u)$ is transitive, $a \subset S(u)$ and $A \cap u \subset u$. (b) holds since $S(u) \supset u$ is transitive and $A \cap u \subset u$. But if we set: $U = \bigcup_{n<\omega}(S^A)^n(u)$, then U is rud closed and $\langle U, A \cap U\rangle$ is amenable. QED (Lemma 2.3.13)

We then set:

Definition 2.3.11.

$$\begin{aligned} &S^A_0 = \emptyset \\ &S^A_{\alpha+1} = S^A(S^A_\alpha) \\ &S^A_\lambda = \bigcup_{\nu<\lambda} S^A_\nu \text{ for limit } \lambda. \end{aligned}$$

We again have: $J_\alpha[A] = S^A_\alpha$ for $\alpha \in \mathrm{Lm}$. A close imitation of the proof of Lemma 2.3.4 gives:

Lemma 2.3.14. $\langle S^A_\nu | \nu < \alpha\rangle$ *is uniformly* $\Sigma_1(J^A_\alpha)$.

Proof: This is exactly as before except that in the formula $\varphi(f)$ we replace $S(f(\nu))$ by $S^A(f(\nu))$. But this is $\Sigma_0(J^A_\alpha)$, since:

$$x \in S^A(u) \leftrightarrow (x \in S(u) \vee x = A \cap u),$$

hence:

$$\begin{aligned} y = &S^A(u) \leftrightarrow \bigwedge z \in y\ z \in S^A(u) \\ &\wedge \bigwedge z \in S(u) z \in y \wedge \bigvee z \in y\ z = A \cap u. \end{aligned}$$

QED (Lemma 2.3.14)

We now show that J_α^A has a uniformly $\Sigma_1(J_\alpha^A)$ well ordering, which we call $<_\alpha^A$ or $<_{J_\alpha^A}$.

Set:

Definition 2.3.12.

$$\begin{aligned} W^A(u,r) = &\{\langle x,y\rangle | \langle x,y\rangle \in W(u,r) \vee \\ &(x \in S(u) \wedge y = A \cap u \notin S(u)\} \end{aligned}$$

If u is transitive and r well orders u, then $W^A(u,r)$ is a well ordering of $S^A(u)$ which end extends r.

We set:

Definition 2.3.13.

$$\begin{aligned} &<_0^A = \emptyset \\ &<_{\nu+1}^A = W^A(S_\nu^A, <_\nu^A) \\ &<_\lambda^A = \bigcup_{\nu<\lambda} <_\nu^A \text{ for limit } < . \end{aligned}$$

Then $<_\nu^A$ is a well ordering of S_ν^A which end extends $<_\xi^A$ for $\xi < \nu$. In particular $<_\alpha^A$ well orders J_α^A for $\alpha \in \Gamma$. We also write: $<_{J_\alpha^A} =: <_\alpha^A$. We set: $<_{L^A} = <_{J^A} = <_\infty^A =: \bigcup_{\nu<\infty} <_\nu^A$.

Just as before we get:

Lemma 2.3.15. *$\langle <_\nu^A | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J_\alpha^A)$.*

The proof is left to the reader. Just as before we get:

Lemma 2.3.16. *$<_\alpha^A$ and $f(u) = \{z | z <_\alpha^A u\}$ are uniformly $\Sigma_1(J_\alpha^A)$.*

Up until now almost everything we proved for the J_α hierarchy could be shown to hold for the J_α^A hierarchy. The condensation lemma, however, is available only in a much weaker form:

Lemma 2.3.17. *Let $X \prec_{\Sigma_1} J_\alpha^A$. Then there are $\overline{\alpha}, \pi, \overline{A}$ such that $\pi : J_{\overline{\alpha}}^{\overline{A}} \overset{\sim}{\longleftrightarrow} X$.*

Proof: By Lemma 2.2.19 there is $\langle \overline{U}, \overline{A}\rangle$ such that $\pi : \langle \overline{U}, \overline{A}\rangle \overset{\sim}{\longleftrightarrow} X$ and $\langle \overline{U}, \overline{A}\rangle$ is rud closed. As before, the condition

$$S^A(f, \nu) \leftrightarrow f = \langle S^A_\xi | \nu < \xi\rangle$$

si Σ_0 in A. Now let $\overline{\nu} < \overline{\alpha}, \pi(\overline{\nu}) = \nu$. As before $f = \langle S_\xi | \xi < \nu\rangle \in X$. Let $\pi(\overline{f} = f$. Then $\overline{f} = \langle S^A_\xi | \xi < \overline{\nu}\rangle$, since $S^{\overline{A}}(\overline{f}, \overline{\nu})$. Then $J^{\overline{A}}_{\overline{\alpha}} \subset \bigcup\limits_{\xi<\overline{\alpha}} S^{\overline{A}}_\xi \subset \overline{U}$. $U \subset J^{\overline{A}}_{\overline{\alpha}}$ then follows as before. QED (Lemma 2.3.17)

A sometimes useful feature of the J^A_α hierarchy is:

Lemma 2.3.18. $x \in J^A_\alpha \to TC(x) \in J^A_\alpha$.

(Hence $\langle TC(x) | x \in J^A_\alpha\rangle$ is $\Pi_1(J^A_\alpha)$ since $u = TC(x)$ is defined by:

$$u \text{ is transitive } \wedge x \subset u \wedge \bigwedge v((v \text{ is transitive } \wedge x \subset v) \to u \subset v)$$

Proof: By induction on α.

Case 1 $\alpha = \omega$ (trivial)

Case 2 $\alpha = \beta + \omega$, $\beta \in \mathrm{Lim}$.
Then every $x \in J^A_\alpha$ has the form $f(\vec{z})$ where $z_1, \ldots, z_n \in J_\beta[A] \cup \{J_\beta[A]\}$ and f is rud in A. By Lemma 2.2.2 we have

$$\bigcup{}^P x \subset \bigcup_{i=1}^n TC(z_i) \subset J_\beta[A] \text{ for some } p < \omega$$

Hence $TC(x) = C_p(x) \cup TC(\bigcup_{i=1}^n TC(z_i)$, where $\langle TC(z) | z \in J_\beta[A]\rangle$ is J^A_β–definable, hence an element of J^A_α.

Case 3 $\alpha \in \mathrm{Lm}^*$ (trivial). QED (Lemma 2.3.18)

Corollary 2.3.19. *If $\alpha \in \mathrm{Lm}^*$, then $\langle TC(x) | x \in J^A_\alpha\rangle$ is uniformly $\Delta_1(J^A_\alpha)$.*

Proof: We have seen that it is $\Pi_1(J^A_\alpha)$. But $TC \restriction J^A_\alpha \in J^A_\alpha$ for all $\beta \in \mathrm{Lm} \cap \alpha$. Hence $u = TC(x)$ is definable in J^A_α by:

$$\begin{aligned}\bigvee f(f \text{ is a function } &\wedge \mathrm{dom}(f) \text{ is transitive } \wedge u = f(x)\\ &\wedge \bigwedge x \in \mathrm{dom}(f) f(x) = x \cup \bigcup f\text{''}nx)\end{aligned}$$

QED (Corollary 2.3.19)