2.7 Liftups

2.7.1 The Σ_0 liftup

A concept which, under a variety of names, is frequently used in set theory is the *liftup* (or as we shall call it here, the Σ_0 *liftup*). We can define it as follows:

Definition 2.7.1. Let M be a J–model. Let $\tau > \omega$ be a cardinal in M. Let $H = H_\tau^M \in M$ and let $\pi : H \to_{\Sigma_0} H'$ cofinally. We say that $\langle M', \pi' \rangle$ is a Σ_0 *liftup* of $\langle M, \pi \rangle$ iff M' is transitive and:

(a) $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$

(b) Every element of M' has the form $\pi'(f)(x)$ for an $x \in H'$ and an $f \in \Gamma^0$, where $\Gamma^0 = \Gamma^0(\tau, M)$ is the set of functions $f \in M$ such that $\text{dom}(f) \in H$.

Note. The condition of being a J–model can be relaxed considerably, but that is uninteresting for our purposes.

Until further notice we shall use the word 'liftup' to mean 'Σ_0 liftup'.

If $\langle M', \pi' \rangle$ is a liftup of $\langle M, \pi \rangle$ it follows easily that:

Lemma 2.7.1. *$\pi' : M \to_{\Sigma_0} M'$ cofinally.*

Proof: Let $y \in M'$, $y = \pi'(f)(x)$ where $x \in H'$ and $f \in \Gamma^0$, then $y \in \pi'(\text{rng}(f))$. QED (Lemma 2.7.1)

Lemma 2.7.2. *$\langle M', \pi' \rangle$ is the only liftup of $\langle M, \pi \rangle$.*

Proof: Suppose not. Let $\langle M^*, \pi^* \rangle$ be another liftup. Let $\varphi(v_1, \ldots, v_n)$ be Σ_0. Then

$$M' \models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_n)(x_n)] \leftrightarrow$$
$$\langle x_1, \ldots, x_n \rangle \in \pi(\{\langle \vec{z} \rangle | M \models \varphi[\vec{f}(\vec{z})]\}) \leftrightarrow$$
$$M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_n)(x_n)].$$

Hence there is an isomorphism σ of M' onto M^* defined by:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x)$$
$$\text{for } f \in \Gamma^0,\ x \in \pi(\text{dom}(f)).$$

But M', M^* are transitive. Hence $\sigma = \text{id},\ M' = M^*,\ \pi' = \pi^*$.

QED (Lemma 2.7.2)

Note. $M \models \varphi[\vec{f}(\vec{z})]$ means the same as

$$\bigvee y_1 \dots y_n (\bigwedge_{i=1}^{n} y_i = f_i(z_i) \wedge M \models \varphi[\vec{y}]).$$

Hence if $e = \{\langle\vec{z}\rangle | M \models \varphi[\vec{f}(\vec{z})]\}$, then $e \subset \overset{n}{\underset{i=1}{\times}} \mathrm{dom}(f_i) \in H$. Hence $e \in M$ by rud closure, since e is $\underline{\Sigma}_0(M)$. But then $e \in H$, since $\mathbb{P}(u) \cap M \subset H$ for $u \in H$.

But when does the liftup exist? In answering this question it is useful to devise a 'term model' for the putative liftup rather like the ultrapower construction:

Definition 2.7.2. Let $M, \tau, \pi : H \to_{\Sigma_0} H'$ be as above. The term model $\mathbb{D} = \mathbb{D}(M, \pi)$ is defined as follows. Let e.g. $M = \langle J_\alpha^A, B\rangle$. $\mathbb{D} =: \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B}\rangle$ where

$D =$ the set of pairs $\langle f, x\rangle$ such that $f \in \Gamma_0$ and $x \in H'$

$$\langle f, x\rangle \cong \langle g, y\rangle \leftrightarrow: \langle x, y\rangle \in \pi(\{\langle z, w\rangle | f(z) = g(y)\})$$
$$\langle f, x\rangle \tilde{\in} \langle g, y\rangle \leftrightarrow: \langle x, y\rangle \in \pi(\{\langle z, w\rangle | f(z) \in g(y)\})$$
$$\tilde{A}\langle f, x\rangle \leftrightarrow: x \in \pi(\{z | Af(z)\})$$
$$\tilde{B}\langle f, x\rangle \leftrightarrow: x \in \pi(\{z | Bf(z)\})$$

Note. $\mathbb{D}$ is an 'equality model', since the identity predicate $=$ is interpreted by $\cong$ rather than the identity.

Łos theorem for $\mathbb{D}$ then reads:

Lemma 2.7.3. *Let $\varphi = \varphi(v_1, \dots, v_n)$ be Σ_0. Then*

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \dots, \langle f_n, x_n\rangle] \leftrightarrow \langle x_1, \dots, x_n\rangle \in \pi(\{\langle\vec{z}\rangle | M \models \varphi[\vec{f}(\vec{z})]\}).$$

Proof: (Sketch)
We prove this by induction on the formula φ. We display a typical case of the induction. Let $\varphi = \bigvee u \in v_1 \Psi$. By bound relettering we can assume *w.l.o.g.* that u is not among $v_1, \dots, v_n$. Hence $u, v_1, \dots, v_n$ is a good sequence for Ψ. We first prove $(\to)$. Assume:

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \dots, \langle f_n, x_n\rangle].$$

Claim $\langle x_1, \dots, x_n\rangle \in \pi(e)$ where

$$e = \{\langle z_1, \dots, z_n\rangle | M \models \varphi[f_1(z_1) \dots f_n(z_n)]\}.$$

Proof: By our assumption there is $\langle g, y\rangle \in D$ such that $\langle g, y\rangle \tilde{\in} \langle f_1, x_1\rangle$ and:

$$\mathbb{D} \models \Psi[\langle g, y\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle].$$

By the induction hypothesis we conclude that $\langle y, \vec{x}\rangle \in \pi(\tilde{e})$ where:

$$\tilde{e} = \{\langle w, \vec{z}\rangle | g(w) \in f_1(z_1) \wedge M \models \Psi[g(w), \vec{f}(\vec{z})\}.$$

Clearly $e, \tilde{e} \in H$ and

$$H \models \bigwedge w, \vec{z}(\langle w, \vec{z}\rangle \in \tilde{e} \to \langle \vec{z}\rangle \in e).$$

Hence

$$H' \models \bigwedge w, \vec{z}(\langle w, \vec{z}\rangle \in \pi(e) \to \langle \vec{z}\rangle \in \pi(e)).$$

Hence $\langle \vec{x}\rangle \in \pi(e)$. QED $(\to)$

We now prove $(\leftarrow)$
We assume that $\langle x_1, \ldots, x_n\rangle \in \pi(e)$ and must prove:

Claim $\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle]$.

Proof: Let $r \in M$ be a well ordering of $\text{rng}(f_1)$. For $\langle \vec{z}\rangle \in e$ set:

$$g(\langle \vec{z}\rangle) = \text{ the } r\text{–least } w \text{ such that } M \models \Psi[w, f_1(z_1), \ldots, f_n(z_n)].$$

Then $g \in M$ and $\text{dom}(g) = e \in H$. Now let $\tilde{e}$ be defined as above with this g. Then:

$$H \models \bigwedge z_1, \ldots, z_n(\langle \vec{z}\rangle \in e \leftrightarrow \langle \langle \vec{z}\rangle, \vec{z}\rangle \in \tilde{e}).$$

But then the corresponding statement holds of $\pi(e), \pi(\tilde{e})$ in H'. Hence

$$\langle \langle \vec{x}\rangle, \vec{x}\rangle \in \pi(\tilde{e}).$$

By the induction hypothesis we conclude:

$$\mathbb{D} \models \Psi[\langle g, \langle \vec{x}\rangle\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle].$$

The conclusion is immediate. QED (Lemma 2.7.3)

The liftup of $\langle M, \pi\rangle$ can only exist if the relation $\tilde{e}$ is well founded:

Lemma 2.7.4. *Let $\tilde{\in}$ be ill founded. Then there is no $\langle M', \pi'\rangle$ such that $\pi' : M \to_{\Sigma_0} M'$. M' is transitive, and $\pi' \supset \pi$.*

Proof: Suppose not. Let $\langle f_{i+1}, x_{i+1}\rangle \tilde{\in} \langle f_i, x_i\rangle$ for $i < w$. Then

$$\langle x_{i+1}, x_i\rangle \in \pi\{\langle z, w\rangle | f_{i+1}(z) \in f_i(w)\}.$$

Hence $\pi'(f_{i+1})(x_{i+1}) \in \pi'(f_i)(x_i)(i < w)$.
Contradiction! QED (Lemma 2.7.4)

Conversely we have:

Lemma 2.7.5. *Let $\tilde{\in}$ be well founded. Then the liftup of $\langle M, \pi\rangle$ exists.*

Proof: We shall explicitly construct a liftup from the term model $\mathbb{D}$. The proof will stretch over several subclaims.

Definition 2.7.3. $x^* = \pi^*(x) =: \langle \text{const}_x, 0\rangle$, where $\text{const}_x =: \{\langle x, 0\rangle\} =$ the constant function x defined on $\{0\}$.

Then:

(1) $\pi^* : M \to_{\Sigma_0} \mathbb{D}$.
Proof: Let $\varphi(v_1, \ldots, v_n)$ be Σ_0. Set:

$$e = \{\langle z_1, \ldots, z_n\rangle | M \models \varphi[\text{const}_{x_1}(z_1), \ldots, \text{const}_{x_n}(z_n)]\}.$$

Obviously:

$$e = \begin{cases} \{\langle 0, \ldots, 0\rangle\} \text{ if } M \models \varphi[x_1, \ldots, x_n] \\ \emptyset \text{ if not.} \end{cases}$$

Hence by Łoz theorem:

$$\begin{aligned} \mathbb{D} \models \varphi[x_1^*, \ldots, x_n^*] &\leftrightarrow \langle 0, \ldots, 0\rangle \in \pi(e) \\ &\leftrightarrow M \models \varphi[x_1, \ldots, x_n] \end{aligned}$$

(2) $\mathbb{D} \models$ Extensionality.
Proof: Let $\varphi(u, v) =: \bigwedge w \in u\, w \in v \ \wedge \bigwedge w \in v\, w \in u$.
Claim $\mathbb{D} \models \varphi[a, b] \to a \cong b$ for $a, b \in \mathbb{D}$. This reduces to the Claim:
Let $a = \langle f, x\rangle, b = \langle g, y\rangle$. Then

$$\begin{aligned} \mathbb{D} \models \varphi[\langle f, x\rangle, \langle g, y\rangle] &\leftrightarrow \langle x, y\rangle \in \pi(e) \\ &\leftrightarrow \langle f, x\rangle \cong \langle g, y\rangle \end{aligned}$$

where

$$\begin{aligned} e &= \{\langle z, w\rangle | M \models \varphi[z, w]\} \\ &= \{\langle z, w\rangle | f(z) = g(w)\} \end{aligned}$$

QED (2)

Since $\cong$ is a congruence relation for $\mathbb{D}$ we can factor $\mathbb{D}$ by $\cong$, getting:

$$\hat{\mathbb{D}} = (\mathbb{D} \setminus \cong) = \langle \hat{D}, \hat{\in}, \hat{A}, \hat{B} \rangle$$

where:

$$\hat{D} = \{\hat{s} | s \in D\}$$
$$\hat{s} =: \{t | t \cong s\} \text{ for } s \in D$$
$$\hat{s} \hat{\in} \hat{t} \leftrightarrow: s \tilde{\in} t$$
$$\hat{A}\hat{s} \leftrightarrow: \tilde{A}s, \hat{B}\hat{s} \leftrightarrow: \tilde{B}s.$$

Then $\hat{\mathbb{D}}$ is a well founded identity model satisfying extensionality. By Mostowski's isomorphism theorem there is an isomorphism k of $\hat{\mathbb{D}}$ onto M', where $M' = \langle |M'|, \in, A', B' \rangle$ is transitive.

Set:

$$[s] =: k(\hat{s}) \text{ for } s \in D$$
$$\pi'(x) =: [x^*] \text{ for } x \in M.$$

Then by (1):

(3) $\pi' : M \to_{\Sigma_0} M'$.

Lemma 2.7.5 will then follow by:

Lemma 2.7.6. *$\langle M', \pi' \rangle$ is the liftup of $\langle M, \pi \rangle$.*

We shall often write $[f, x]$ for $[\langle f, x \rangle]$. Clearly every $s \in M'$ has the form $[f, x]$ where $f \in M$; $\text{dom}(f) \in H$, $x \in H'$.

Definition 2.7.4. $\tilde{H}$ =: the set of $[f, x]$ such that $\langle f, x \rangle \in D$ and $f \in H$.

We intend to show that $[f, x] = \pi(f)(x)$ for $x \in \tilde{H}$. As a first step we show:

(4) $\tilde{H}$ is transitive.

Proof: Let $s \in [f, x]$ where $f \in H$.

Claim $s = [g, y]$ for a $g \in H$.

Proof: Let $s = [g', y]$. Then $\langle y, x \rangle \in \pi(e)$ where: $e = \{\langle u, v \rangle | g'(u) \in f(v)\}$ set:

$$e' = \{u | g'(u) \in \text{rng}(f)\}, \ g = g' \upharpoonright e'.$$

Then $g \subset \text{rng}(f) \times \text{dom}(g') \in H$. Hence $g \in H$. Then $[g', y] = [g, y]$ since $\pi(g')(y) = \pi(g)(y)$ and hence
$\langle y, y \rangle \in \pi(\{\langle u, v \rangle | g'(u) = g(v)\})$. But $e = \{\langle u, v \rangle | g(u) \in f(v)\}$. Hence $[g, y] \in [f, x]$. QED (4)

But then:

(5) $[f,x] = \pi(f)(x)$ for $f \in H, \langle f,x\rangle \in D$.

Proof: Let $f,g \in H, \langle f,x\rangle, \langle g,y\rangle \in D$. Then:

$$\begin{aligned}[f,x] \in [g,y] &\leftrightarrow \langle x,y\rangle \in \pi(e)\\ &\leftrightarrow \pi(f)(x) \in \pi(g)(y)\end{aligned}$$

where $e = \{\langle u,v\rangle | f(u) \in g(v)\}$. Hence there is an $\in$–isomorphism σ of H onto $\tilde{H}$ defined by:

$$\sigma(\pi(f)(x)) =: [f,x].$$

But then $\sigma = \text{id}$, since $H, \tilde{H}$ are transitive. (5)

But then:

(6) $\pi' \supset \pi$.

Proof: Let $x \in H$. Then $\pi'(x) = [\text{const}_x, 0] = \pi(\text{const}_x)(0) = \pi(x)$ by (5).

(7) $[f,x] = \pi'(f)(x)$ for $\langle f,x\rangle \in D$.

Proof: Let $a = \text{dom}(f)$. Then $[\text{id}_a, x] = \text{id}_{\pi(a)}(x) = x$ by (5). Hence it suffices to show:

$$[f,x] = [\text{const}_f, 0]([\text{id}_a, x]).$$

But this says that $\langle x,0\rangle \in \pi(e)$ where:

$$\begin{aligned} e &= \{\langle z,u\rangle | f(z) = \text{const}_f(u)(\text{id}_a(z))\}\\ &= \{\langle z,0\rangle | f(z) = f(z)\} = a \times \{0\}.\end{aligned}$$

QED (7)

Lemma 2.7.6 is then immediate by (3), (6) and (7). QED (Lemma 2.7.6)

Lemma 2.7.7. *Let $\pi^* \supset \pi$ such that $\pi^* : M \to_{\Sigma_0} M^*$. Then the liftup $\langle M', \pi'\rangle$ of $\langle M, \pi\rangle$ exists. Moreover there is a $\sigma : M' \to_{\Sigma_0} M^*$ uniquely defined by the condition:*

$$\sigma \restriction H' = \text{id},\ \sigma\pi' = \pi^*.$$

Proof: $\langle M', \pi'\rangle$ exists, since $\tilde{\in}$ is well founded, since $\langle f,x\rangle \tilde{\in} \langle g,y\rangle \leftrightarrow \pi^*(f)(x) \in \pi^*(g)(y)$. But then:

$$\begin{aligned} M' &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \leftrightarrow\\ &\leftrightarrow \langle x_1, \ldots, x_r\rangle \in \pi(e)\\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)]\end{aligned}$$

where $e = \{\langle z_1, \ldots, z_r\rangle | M \models \varphi[\vec{f}(\vec{z})]\}$. Hence there is $\sigma : M' \to_{\Sigma_0} M^*$ defined by:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x) \text{ for } \langle f, x\rangle \in D.$$

Now let $\tilde{\sigma} : M' \to_{\Sigma_0} M^*$ such that $\tilde{\sigma} \upharpoonright H' = \text{id}$ and $\tilde{\sigma}\pi' = \pi^r$.

Claim $\tilde{\sigma} = \sigma$.
Let $s \in M'$, $s = \pi'(f)(x)$. Then $\tilde{\sigma}(\pi'(f)) = \pi^*(f)$, $\tilde{\sigma}(x) = x$. Hence $\tilde{\sigma}(s) = \pi^*(f)(x) = \sigma(s)$. QED (Lemma 2.7.7)

2.7.2 The $\Sigma_0^{(n)}$ liftup

From now on suppose M to be acceptable. We now attempt to generalize the notion of Σ_0 liftup. We suppose as before that $\tau > w$ is a cardinal in M and $H = H_\tau^M$. As before we suppose that $\pi' : H \to_{\Sigma_0} H'$ cofinally. Now let $\rho^n \geq \tau$. The Σ_0–liftup was the "minimal" $\langle M', \pi'\rangle$ such that $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$. We shall now consider pairs $\langle M', \pi'\rangle$ such that $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0^n} M'$. Among such pairs $\langle M', \pi'\rangle$ we want to define a "minimal" one and show, if possible, that it exists. The minimality of the Σ_0 liftup was expressed by the condition that every element of M' have the form $\pi'(f)(x)$, where $x \in H'$ and $f \in \Gamma^0(\tau, M)$. As a first step to generalizing this definition we replace $\Gamma^0(\tau, M)$ by a larger class of functions $\Gamma^n(\tau, M)$.

Definition 2.7.5. Let $n > 0$ such that $\tau \leq \rho_M^n$. $\Gamma^n = \Gamma^n(\tau, M)$ is the set of maps f such that

(a) $\text{dom}(f) \in H$

(b) For some $i < n$ there is a good $\Sigma_1^{(i)}(M)$ function G and a parameter $p \in M$ such that $f(x) = G(x, p)$ for all $x \in \text{dom}(f)$.

Note. Good $\Sigma_1^{(i)}$ functions are many sorted, hence any such function can be identified with a pair consisting of its field and its arity. An element of Γ^n, on the other hand, is 1–sorted in the classical sense, and can be identified with its field.

Note. This definition makes sense for the case $n = \omega$, and we will not exclude this case. A $\Sigma_0^{(\omega)}$ formula (or relation) then means any formula (or relation) which is $\Sigma_0^{(i)}$ for an $i < \omega$ — i.e. $\Sigma_0^{(\omega)} = \Sigma^*$.

We note:

Lemma 2.7.8. *Let $f \in \Gamma^n$ such that $\text{rng}(f) \subset H^i$, where $i < n$. Then $f(x) = G(x, p)$ for $x \in \text{dom}(f)$ where G is a good $\Sigma_1^{(h)}$ function to H^i for some $h < n$.*

Proof: Let $f(x) = G'(x,p)$ for $x \in \mathrm{dom}(f)$ where G' is a good $\Sigma_1^{(h)}$ function to H^j where $h, j < n$. Since every good $\Sigma_1^{(h)}$ function is a good Σ_1^k function for $k \geq h$, we can assume *w.l.o.g.* that $i, j \leq h$. Let F be the identity function defined by $v^i = u^j$ (i.e. $y^i = F(x^j) \leftrightarrow y^i = x^j$). Set: $G(x,y) \simeq: F(G'(x,y))$. Then F is a good $\Sigma_1^{(h)}$ function and so is G, where $f(x) = G(x,p)$ for $x \in \mathrm{dom}(f)$.

QED (Lemma 2.7.8)

Lemma 2.7.9. $\Gamma^i(\tau, M) \subset \Gamma^n(\tau, M)$ *for* $i < n$*.*

Proof: For $0 < i$ this is immediat by the definition. Now let $i = 0$. If $f \in \Gamma^0$, then $f(x) = G(x,f)$ for $x \in \mathrm{dom}(f)$ where G is the $\Sigma_0^{(0)}$ function defined by

$$y = G(x,f) \leftrightarrow: \ (f \text{ is a function} \land \land \langle y, x\rangle \in f).$$

QED (Lemma 2.7.9)

The "natural" minimality condition for the $\Sigma_0^{(n)}$ liftup would then read: Each element of M has the form $\pi'(f)(x)$ where $x \in H'$ and $f \in \Gamma^n$. But what sense can we make of the expression "$\pi'(f)(x)$" when f is not an element of M? The following lemma rushes to our aid:

Lemma 2.7.10. *Let* $\pi' : M \to_{\Sigma_0^{(n)}} M'$ *where* $n > 0$ *and* $\pi' \supset \pi$*. There is a unique map* π'' *on* $\Gamma^n(\tau, M)$ *with the following property:*

* *Let* $f \in \Gamma^n(\tau, M)$ *such that* $f(x) = G(x,p)$ *for* $x \in \mathrm{dom}(f)$ *where* G *is a good* $\Sigma_1^{(i)}$ *function for an* $i < n$ *and* χ *is a good* $\Sigma_1^{(i)}$ *definition of* G*. Let* G' *be the function defined on* M' *by* χ*. Let* $f' = \pi''(f)$*. Then* $\mathrm{dom}(f') = \pi(\mathrm{dom}(f))$ *and* $f'(x) = G'(x, \pi'(p))$ *for* $x \in \mathrm{dom}(f')$*.*

Proof: As a first approximation, we simply pick G, χ with the above properties. Let G' then be as above. Let $d = \mathrm{dom}(f)$. The statement $\bigwedge x \in d \bigvee y\, y = G(x,p)$ is $\Sigma_0^{(n)}$ is d, p, so we have:

$$\bigwedge x \in \pi(d) \bigvee y\, y = G'(x, \pi(p)).$$

Define f_0 by $\mathrm{dom}(f_0) = \pi(d)$ and $f_0(x) = G'(x, \pi(p))$ for $x \in \pi(d)$. The problem is, of course, that G, χ were picked arbitrarily. We might also have:

$$f(x) = H(x,q) \text{ for } x \in d,$$

where H is $\Sigma_1^{(j)}(M)$ for a $j < n$ and Ψ is a good $\Sigma_1^{(j)}$ definition of H. Let H' be the good function on M' defined by Ψ. As before we can define f_1

by $\text{dom}(f_1) = \pi(d)$ and $f_1(x) = H'(x, \pi'(q))$ for $x \in \pi(d)$. We must show: $f_0 = f_1$. We note that:

$$\bigwedge x \in dG(x,p) = H(x,q).$$

But this is a $\Sigma_0^{(n)}$ statement. Hence

$$\bigwedge x \in \pi(d)G'(x,p) = H'(x,q).$$

Then $f_0 = f_1$. QED (Lemma 2.7.10)

Moreover, we get:

Lemma 2.7.11. *Let $n, \pi, \tau, \pi', \pi''$ be as above. Then $\pi''(f) = \pi'(f)$ for $f \in \Gamma^0(\tau, M)$.*

Proof: We know $f(x) = G(x, f)$ for $x \in d = \text{dom}(f)$, where:

$$y = G(x,f) \leftrightarrow: (f \text{ is a function } \wedge y = f(x)).$$

Then $\pi''(f)(x) = G'(x, \pi'(f)) = \pi'(f)(x)$ for $x \in \pi(d)$, where G' has the same definition over M'. QED (Lemma 2.7.11)

Thus there is no ambiguity in writing $\pi'(f)$ instead of $\pi''(f)$ for $f \in \Gamma^n$. Doing so, we define:

Definition 2.7.6. Let $\omega < \tau < \rho_M^n$ where $n \leq \omega$ and τ is a cardinal in M. Let $H = H_\tau^M$ and let $\pi : H \to_{\Sigma_0} H'$ cofinally. We call $\langle M', \pi' \rangle$ a $\Sigma_0^{(n)}$ *liftup* of $\langle M, \pi \rangle$ iff the following hold:

(a) $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0^{(n)}} M'$.

(b) Each element of M' has the form $\pi'(f)(x)$, where $f \in \Gamma^n(\tau, M)$ and $x \in H'$.

(Thus the old Σ_0 liftup is simply the special case: $n = 0$.)

Definition 2.7.7. $\Gamma_i^n(\tau, M)$ =: the set of $f \in \Gamma^n(\tau, M)$ such that either $i < n$ and $\text{rng}(f) \subset H_M^i$ or $i = n < \omega$ and $f \in H_M^i$.

(Here, as usual, $H^i = J_{\rho_M^i}[A]$ where $M = \langle J_\alpha^A, B \rangle$.)

Lemma 2.7.12. *Let $f \in \Gamma_i^n(\tau, M)$. Let $\pi' : M \to_{\Sigma_0^{(n)}} M'$ where $\pi' \supset \pi$. Then $\pi'(f) \in \Gamma_i^n(\pi'(\tau), M')$.*

Proof:

Case 1 $i = n$. Then $f \in H^M_{\rho^n_M}$. Hence $\pi'(f) \in H^{M'}_{\rho^n_M}$.

Case 2 $i < n$.

By Lemma 2.7.9 for some $h < n$ there is a good $\Sigma_1^{(n)}(M)$ function $G(u, v)$ to H^i and a parameter p such that

$$f(x) = G(x, p) \text{ for } x \in \text{dom}(f).$$

Hence:

$$\pi'(f)(x) = G'(x, \pi'(p)) \text{ for } x \in \text{dom}(\pi(f)),$$

where G' is defined over M' by the same good $\Sigma_1^{(n)}$ definition. Hence $\text{rng}(\pi'(f)) \subset H^i_{M}$. QED (Lemma 2.7.12)

The following lemma will become our main tool in understanding $\Sigma_0^{(n)}$ liftups.

Lemma 2.7.13. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $f_l \in \Gamma^n_{i_l}(l = 1, \ldots, r)$. Then:*

(a) The relation P is $\Sigma_0^{(n)}$ in a parameter p where:

$$P(\vec{z}) \leftrightarrow: R(f_1(z_1), \ldots, f_r(z_r)).$$

(b) Let $\pi' \supset \pi$ such that $\pi' : M \to_{\Sigma_0^{(n)}} M'$. Let R' be $\Sigma_0^{(n)}(M')$ by the same definition as R. Then P' is $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition as P in p, where:

$$P'(\vec{z}) \leftrightarrow: R'(\pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)).$$

Before proving this lemma we note some corollaries:

Corollary 2.7.14. *Let $e = \{\langle\vec{z}\rangle | P(\vec{z})\}$. Then $e \in H$ and $\pi(e) = \{\langle\vec{z}\rangle | P'(\vec{z})\}$.*

Proof: Clearly $e \subset d = \underset{l=1}{\overset{r}{\times}} \text{dom}(f_l) \in H$. But then $d \in H_{\rho^n}$ and $e \in H_{\rho^n}$ since $\langle H_{\rho^n}, P \cap H_{\rho^n}\rangle$ is amenable. Hence $e \in H$, since $H = H^M_\tau$ and therefore $\mathbb{P}(u) \cap M \subset H$ for $u \in H$.

Now set $e' = \{\langle\vec{z}\rangle | P'(\vec{z})\}$. Then $e' \subset \pi(d) = \underset{l=1}{\overset{r}{\times}} \text{dom}(\pi(f_l))$ since $\pi' \supset \pi$ and hence $\pi(\text{dom}(f_l)) = \text{dom}(\pi(f_l))$. But

$$\bigwedge \langle\vec{z}\rangle \in d(\langle\vec{z}\rangle \in e \leftrightarrow P(\vec{z}))$$

which is a $\Sigma_0^{(n)}$ statement about e, p. Hence the same statement holds of $\pi(e), \pi(p)$ in M'. Hence

$$\bigwedge \langle \vec{z} \rangle \in \pi(d)(\langle \vec{z} \rangle \in \pi(e) \leftrightarrow P'(\vec{z})).$$

Hence $\pi(e) = e'$. QED (Corollary 2.7.14)

Corollary 2.7.15. *$\langle M, \pi \rangle$ has at most one $\Sigma_0^{(n)}$ liftup $\langle M', \pi' \rangle$.*

Proof: Let $\langle M^*, \pi^* \rangle$ be a second such. Let $\varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ be a $\Sigma_0^{(n)}$ formula. (In fact, we could take it here as being $\Sigma_0^{(0)}$.) Let $e = \{\langle \vec{z} \rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$ where $f_l \in \Gamma_{i_l}^n (l = 1, \ldots, r)$. Then:

$$\begin{aligned} M' &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \leftrightarrow \\ &\leftrightarrow \langle x_1, \ldots, x_r \rangle \in \pi(e) \\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)] \end{aligned}$$

for $x_l \in \pi(\text{dom}(f_l)(l = 1, \ldots, r)$.

Hence there is an isomorphism $\sigma : M' \tilde{\to} M^*$ defined by:

$$\sigma(\pi'(f)(x)) =: \pi^*(f)(x)$$

for $f \in \Gamma^n$, $x \in \pi(\text{dom}(f))$. But M', M^* are transitive. Hence $\sigma = \text{id}, M' = M^*, \pi' = \pi^*$. QED (Corollary 2.7.15)

We now prove Lemma 2.7.13 by induction on n.

Case 1 $n = 0$.

Then $f_1, \ldots, f_r \in M$ and P is Σ_0 in $p = \langle f_1, \ldots, f_r \rangle$, since f_i is rudimentary in p and for sufficiently large h we have:

$$P(\vec{z}) \leftrightarrow \bigvee y_1, \ldots, y_r \in C_h(p)(\bigwedge_{i=1}^{r} y_i = f_i(\vec{z}_i) \wedge R(\vec{y}))$$

where R is Σ_0. If P' has the same Σ_0 definition over M' in $\pi'(p)$, then

$$\begin{aligned} P'(z) &\leftrightarrow \bigvee_{y_1, \ldots, y_r} \in C_h(\pi(p))(\bigwedge_{n=1}^{r} y_i = \pi(f_i)(z_i) \wedge R(\vec{y})) \\ &\leftrightarrow R(\pi(\vec{f})(\vec{z})) \end{aligned}$$

QED

Case 2 $n = \omega$.

Then $\Sigma_0^\omega = \bigcup_{h<w} \Sigma_1^{(h)}$. Let $R(x_1^{i_1}, \ldots, x_r^{l_r})$ be $\Sigma_1^{(h)}$. Since every $\Sigma_1^{(h)}$ relation is $\Sigma_1^{(k)}$ for $k \geq h$, we can assume h taken large enough that $i_1, \ldots, i_r \leq h$. We can also choose it large enough that:

$$f_l(z) \simeq G_l(z, p) \text{ for } l = 1, \ldots, v$$

where G_l is a good $\Sigma_1^{(h)}$ map to H^{i_l}. (We assume *w.l.o.g.* that p is the same for $l = 1, \ldots, r$ and that $d_l = \text{dom}(f_l)$ is rudimentary in p.) Set:

$$P(\vec{z}, y) \leftrightarrow: R(G_1 x_1, y), \ldots, G(x_r, y)).$$

By §6 Lemma 2.6.24, P is $\Sigma_1^{(h)}$ (uniformly in the $\Sigma_1^{(h)}$ definition of R and $G_1, \ldots, G_r$). Moreover:

$$P(\vec{z}) \leftrightarrow P(\vec{z}, p).$$

Thus P is uniformly $\Sigma_1^{(h)}$ in p, which proves (a). But letting P' have the same $\Sigma_1^{(h)}$ definition in $\pi'(p)$ over M', we have:

$$\begin{aligned} P'(\vec{z}) &\leftrightarrow P'(\vec{z}, \pi'(p)) \\ &\leftrightarrow R'(\pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)), \end{aligned}$$

which proves (b). QED (Case 2)

Case 3 $0 < n < w$.

Let $n = m+1$. Rearranging arguments as necessary, we can take R as given in the form:

$$R(y_1^n, \ldots, y_s^n, x_1^{i_1}, \ldots, x_r^{i_r})$$

where $i_1, \ldots, i_r \leq m$. Let $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$ and let $g_1, \ldots, g_1 \in \Gamma_n^n$.

Claim

(a) P is $\Sigma_0^{(n)}$ in a parameter p where

$$P(\vec{w}, \vec{z}) \leftrightarrow: R(\vec{g}(\vec{w}), \vec{f}(\vec{z})).$$

(b) If π', M' are as above and P' is $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition, then

$$P'(w, \vec{z}) \leftrightarrow R'(\pi'(\vec{g})(\vec{w}), \pi'(\vec{f})(\vec{z}))$$

where R' has the same $\Sigma_0^{(n)}$ definition over M'.

We prove this by first substituting $\vec{f}(\vec{z})$ and then $\vec{g}(\vec{w})$, using two different arguments. The claim then follows from the pair of claims:

Claim 1 Let:

$$P_0(\vec{y}^n, \vec{z}) \leftrightarrow= R(y^n, f_1(z_1), \dots, f_r(z_r)).$$

Then:

(a) P_0 is $\Sigma_0^{(n)}(M)$ in a parameter p_0.

(b) Let π', M', R' be as above. Let P'_0 have the same $\Sigma_0^{(n)}(M')$ definition in $\pi'(p_0)$. Then:

$$P'_0(\vec{y}^n, \vec{z}) \leftrightarrow R'(y^n, \pi'(\vec{f})(\vec{z})).$$

Claim 2 Let

$$P(\vec{w}, \vec{z}) \leftrightarrow: P_0(g_1(w_1), \dots, g_s(w_s), \vec{z}).$$

Then:

(a) P is $\Sigma_0^{(n)}(M)$ in a parameter p.

(b) Let π', M', P'_0 be as above. Let P' have the same $\Sigma_1^{(n)}(M')$ definition in $\pi'(p)$. Then

$$P'(\vec{w}, \vec{z}) \leftrightarrow P'_0(\pi'(\vec{g})(\vec{w}), \vec{z}).$$

We prove Claim 1 by imitating the argument in Case 2, taking $h = m$ and using §6 Lemma 2.6.11. The details are left to the reader. We then prove Claim 2 by imitating the argument in Case 1: We know that $g_1, \dots, g_s \in H^n$. Set: $p = \langle g_1, \dots, g_n, p \rangle$. Then P is $\Sigma_0^{(n)}(M)$ in p, since:

$$P(\vec{w}, \vec{z}) \leftrightarrow \bigvee y_1 \dots y_s \in C_h(p)(\bigwedge_{i=1}^{s} y_i = g_i(w_i) \wedge P_0(\vec{y}, \vec{z}))$$

where g_i, p_0 are rud in P, for a sufficiently large h. But if P' is $\Sigma_0^{(n)}(M')$ in $\Pi'(P)$ by the same definition, we obviously have:

$$\begin{aligned} P'(\vec{w}, \vec{z}) \quad & \leftrightarrow \bigvee y_1 \dots y_r(\bigwedge_{i=1}^{s} y_i = \pi'(g)(w_i) \wedge P'_0(\vec{y}, \vec{z})) \\ & P'_0(\pi'(\vec{g})(\vec{w}), \vec{z}). \end{aligned}$$

QED (Lemma 2.7.13)

We can repeat the proof in Case 3 with "extra" arguments $\vec{u}^n$. Thus, after rearranging arguments we would have $R(\vec{u}^n, \vec{y}^n, x_1^{i_1}, \dots, x_r^{i_r})$ where $i_1, \dots, i_r < n$. We would then define

$$P(\vec{u}^n, \vec{w}, \vec{z}) \leftrightarrow: R(\vec{u}^n, \vec{g}(\vec{w}), \vec{f}(\vec{z})).$$

This gives us:

Corollary 2.7.16. *Let $n < w$. Let $R(\vec{u}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_p \leq n$. Let $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$. Set:*

$$P(\vec{u}^n, \vec{z}) \leftrightarrow: R(\vec{u}^n, f_1(z_1), \ldots, f_r(z_r)).$$

Then:

(a) $P(\vec{u}^n, \vec{z})$ is $\Sigma_0^{(n)}$ in a parameter p.

(b) Let $\pi' \supset \pi$ such that $\pi' : M \to_{\Sigma_0^{(n)}} M'$. Let R' be $\Sigma_0^{(n)}(M')$ by the same definition. Let P' be $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition. Then

$$P'(\vec{u}^n, \vec{z}) \leftrightarrow R'(\vec{u}^n, \pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)).$$

By Corollary 2.7.15 $\langle M, \pi \rangle$ can have at most one $\Sigma_0^{(n)}$ liftup. But when does it have a liftup? In order to answer this — as before — define a term model $\mathbb{D} = \mathbb{D}^{(n)}$ for the supposed liftup, which will then exist whenever $\mathbb{D}$ is well founded.

Definition 2.7.8. Let M, τ, H, H', π be as above where $\rho_M^n \geq \tau, n \leq w$. The $\Sigma_0^{(n)}$ *term model* $\mathbb{D} = \mathbb{D}^{(n)}$ is defined as follows: (Let e.g. $M = \langle J_\alpha^A, B \rangle$.) We set: $\mathbb{D} = \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B} \rangle$ where:

$$\begin{aligned} D = D^{(n)} =: \ & \text{the set of pairs } \langle f, x \rangle \\ & \text{such that } f \in \Gamma^n(\tau, M) \text{ and} \\ & x \in \pi(\text{dom}(f)) \end{aligned}$$

$\langle f, x \rangle \cong \langle g, y \rangle \leftrightarrow: \langle x, y \rangle \in \pi(e)$, where

$$e = \{\langle z, w \rangle | f(z) = g(w)\}.$$

$\langle f, x \rangle \tilde{\in} \langle g, y \rangle \leftrightarrow: \langle x, y \rangle \in \pi(e)$, where

$$e = \{\langle z, w \rangle | f(z) \in g(w)\}$$

(similarly for $\tilde{A}, \tilde{B}$).

We shall interpret the model $\mathbb{D}$ in a many sorted language with variables of type $i < \omega$ if $n = \omega$ and otherwise of type $i \leq n$. The variables v^i will range over the domain D_i defined by:

Definition 2.7.9. $D_i = D_i^{(n)} =: \{\langle f, x \rangle \in D | f \in \Gamma_i^n\}$.

Under this interpretation we obtain Łos theorem in the form:

Lemma 2.7.17. *Let* $\varphi(v_1^{i_1},\dots,v_r^{i_r})$ *be* $\Sigma_0^{(n)}$. *Then:*

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle,\dots,\langle f_r,x_r\rangle] \leftrightarrow \langle x_1,\dots,x_r\rangle \in \pi(e)$$

where $e = \{\langle\vec{z}\rangle | M \models \varphi[f_1(z_1),\dots,f_r(z_r)]\}$ *and* $\langle f_l, x_l\rangle \in D_{i_l}$ *for* $l = 1,\dots,r$.

Proof: By induction on i we show:

Claim If $i < n$ or $i = n < \omega$, then the assertion holds for $\Sigma_0^{(i)}$ formulae.

Proof: Let it hold for $j < i$. We proceed by induction on the formula φ.

Case 1 φ is primitive (i.e. φ is $v_i \dot{\in} v_j, v_i \dot{=} v_j, \dot{A}v_i$ or $\dot{B}v_i$ (for $M = \langle J_\alpha^A, B\rangle$). This is immediate by the definition of $\mathbb{D}$.

Case 2 φ is $\Sigma_h^{(j)}$ where $j < i$ and $h = 0$ or 1. If $h = 0$ this is immediate by the induction hypothesis. Let $h = 1$. Then $\varphi = \bigvee u^j \Psi$, where Ψ is $\Sigma_0^{(i)}$. By bound relettering we can assume *w.l.o.g.* that u^i is not in our good sequence $v_1^{i_1},\dots,v_r^{i_r}$. We prove both directions, starting with $(\rightarrow)$:

Let $\mathbb{D} \models \varphi[\langle f_1,x_1\rangle,\dots,\langle f_r,x_r\rangle]$. Then there is $\langle g,y\rangle \in D_j$ such that

$$\mathbb{D} \models \Psi[\langle g,y\rangle,\langle f_1,x_1\rangle,\dots,\langle f_r,x_r\rangle]$$

($u^j, \vec{v}$ being the good sequence for Ψ). Set $e' = \{\langle w,\vec{z}\rangle | M \models \Psi[g(w),\vec{z}(\vec{x})]\}$. Then $\langle y,\vec{x}\rangle \in \pi(e')$ by the induction hypothesis on i. But in M we have:

$$\bigwedge w,\vec{z}(\langle w,\vec{z}\rangle \in e' \rightarrow \langle\vec{z}\rangle \in e).$$

This is a Π_1 statement about e', e. Since $\pi : H \rightarrow_{\Sigma_1} H'$ we can conclude:

$$\bigwedge w,\vec{z}(\langle w,\vec{z}\rangle \in \pi(e') \rightarrow \langle\vec{z} \in \pi(e)).$$

But $\langle y,\vec{x}\rangle \in \pi(e')$ by the induction hypothesis. Hence $\langle\vec{x} \in \pi(e)$. This proves $(\rightarrow)$. We now prove $(\leftarrow)$. Let $\langle\vec{x}\rangle \in \pi(e)$. Let R be the $\Sigma_0^{(j)}$ relation

$$R(w,z_1,\dots,z_r) \leftrightarrow= M \models \varphi[w,z_1,\dots,z_r].$$

Let G be a $\Sigma_0^{(j)}(M)$ map to H^j which uniformizes R. Then G is a spezialization of a function $G'(z_1^{h_1},\dots,z_r^{h_r})$ such that $h_l \le j$ for $l \le j$. Thus G' is a good $\Sigma_0^{(j)}$ function. But

$$f_l(z) = F_l(z,p) \text{ for } z \in \mathrm{dom}(f_l) \text{ for } l = 1,\dots,r$$

where F_l is a good $\Sigma_0^{(k)}$ map to H^{h_l} for $l = 1, \ldots, r$ and $j \leq k < i$. (We assume *w.l.o.g.* that the parameter p is the same for all $l = 1, \ldots, r_n$.) Define $G''(u^k, w)$ by:

$$G''(u, w) \simeq: G'((u)_0^{r-1}, \ldots, (u)_{r-1}^{r-1}, w)$$

then G'' is a good $\Sigma_1^{(k)}$ function. Define g by: $\mathrm{dom}(g) = \underset{i=1}{\overset{r}{\times}} \mathrm{dom}(f_i)$ and: $g(\langle\vec{z}\rangle) = G''(\langle\vec{z}\rangle, p)$ for $\langle\vec{z}\rangle \in \mathrm{dom}(g)$. Then $g \in \Gamma^n$ and $g(\langle\vec{z}\rangle) = G(f_1(z_1), \ldots, f_r(z_r))$. Hence, letting:

$$e' = \{\langle w, \vec{z}\rangle | M \models \Psi[g(w), \vec{f}(\vec{z})]\},$$

we have:

$$\bigwedge \vec{z}(\langle\vec{z}\rangle \in e \leftrightarrow \langle\langle\vec{z}\rangle, \vec{z}\rangle \in e').$$

This is a Π_1 statement about e, e' in H. Hence in H' we have:

$$\bigwedge \vec{z}(\langle\vec{z}\rangle \in \pi(e) \leftrightarrow \langle\langle\vec{z}\rangle, \vec{z}\rangle \in \pi(e')).$$

But then $\langle\langle\vec{z}\rangle, \vec{z}\rangle \in \pi(e')$. By the induction hypothesis we conclude:

$$\mathbb{D} \models \Psi[\langle g, \langle\vec{z}\rangle\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_r, x_r\rangle].$$

Hence:

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_r, x_r\rangle].$$

QED (Case 2)

Case 3 φ is $\Psi_0 \wedge \Psi_1, \Psi_0 \wedge \Psi_1, \Psi_0 \rightarrow \Psi_1, \Psi_0 \leftrightarrow \Psi_1$, or $\neg\Psi$.

This is straightforward and we leave it to the reader.

Case 4 $\varphi = \bigvee u^i \in v_l \chi$ or $\bigwedge u^i \in v_l \chi$, where v_l has type $\geq i$. We display the proof for the case $\varphi = \bigvee u^i \in v_l \chi$. We again assume *w.l.o.g.* that $u' \neq v_j$ for $j = 1, \ldots, r$. Set: $\Psi = (u^i \in v_l \wedge \chi)$. Then φ is equivalent to $\bigvee u^i \Psi$. Using the induction hypothesis for χ we easily get:

$$(*) \quad \begin{aligned} \mathbb{D} \models \quad & \Psi[\langle g, y\rangle, \langle f_1, x_i\rangle, \ldots, \langle f_r, x_r\rangle] \leftrightarrow \\ & \langle y, x_1, \ldots, x_n\rangle \in \pi(e') \end{aligned}$$

where $e' = \{\langle w, \vec{z}\rangle | M \models \Psi[g(w), \vec{f}(\vec{z})]\}$. Using $(*)$, we consider two subcases:

Case 4.1 $i < n$.
We simply repeat the proof in Case 2, using $(*)$ and with i in place of j.

Case 4.2 $i = n < w$.

(Hence v_l has type n.) For the direction $(\rightarrow)$ we can again repeat the proof in Case 2. For the other direction we essentially revert to the proof used initially for Σ_0 liftups.

We know that $e \in H$ and $\langle \vec{x} \rangle \in \pi(e)$, where $e = \{\langle \vec{z} \rangle | M \models \varphi[f_1(z_1), \dots, f_r(z_r)]\}$. Set:

$$R(w^n, \vec{z}) \leftrightarrow: M \models \Psi[w^n, f_1(z_1), \dots, f_r(z_r)].$$

Then R is $\underline{\Sigma}_0^{(n)}$ by Corollary 2.7.16. Moreover $\bigvee w^n R(w^n, \vec{z}) \leftrightarrow \langle \vec{z} \rangle \in e$. Clearly $f_l \in H_M^n$ since $f_l \in \Gamma_n^n$. Let $s \in H_M^n$ be a well odering of $\bigcup \mathrm{rng}(f_l)$. Clearly:

$$\begin{aligned} R(w^n, \vec{z}) \quad &\rightarrow w^n \in f_l(z_l) \\ &\rightarrow w^n \in \bigcup \mathrm{rng}(f_l). \end{aligned}$$

We define a function g with domain e by:

$$g(\langle \vec{z} \rangle) = \text{the } s\text{–least } w \text{ such that } R(w, \vec{z}).$$

Since R is $\underline{\Sigma}_0^{(n)}$, it follows easily that $g \in H_{\rho^n}^M$. Hence $g \in \Gamma_n^n$. But then

$$\bigwedge \vec{z} (\langle \vec{z} \rangle \in e \leftrightarrow \langle \langle \vec{z} \rangle, \vec{z} \rangle \in e'), \text{ where } e' \text{ is defined as above, using this } g.$$

Hence in H' we have:

$$\bigwedge \vec{z} (\langle \vec{z} \rangle \in \pi(e) \leftrightarrow \langle \langle \vec{z} \rangle, \vec{z} \rangle \in \pi(e')).$$

Since $\langle \vec{x} \rangle \in \pi(e)$ we conclude that $\langle \langle \vec{x} \rangle, \vec{x} \rangle \in \pi(e')$. Hence:

$$\mathbb{D} \models \Psi[\langle g, \langle \vec{x} \rangle \rangle, \langle f_1, x_1 \rangle, \dots, \langle f_r, x_r \rangle].$$

Hence:

$$\mathbb{D} \models \varphi[\langle f_1, x_1 \rangle, \dots, \langle f_r, x_r \rangle].$$

QED (Lemma 2.7.17)

Exactly as before we get:

Lemma 2.7.18. *If $\tilde{\in}$ is ill founded, then the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi \rangle$ does not exist.*

We leave it to the reader and prove the converse:

Lemma 2.7.19. *If $\tilde{\in}$ is well founded, then the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi \rangle$ exists.*

Proof: We shall again use the term model $\mathbb{D}$ to define an explicit $\Sigma_0^{(n)}$ liftup. We again define:

Definition 2.7.10. $x^* = \pi^*(x) =: \langle \text{const}_x, 0\rangle$, where $\text{const}_x =: \{\langle x, 0\rangle\} =$ the constant function x defined on $\{0\}$.

Using Łos theorem Lemma 2.7.17 we get:

(1) $\pi^* : M \rightarrow_{\Sigma_0^{(n)}} \mathbb{D}$
(where the variables v^i range over D_i on the $\mathbb{D}$ side).

The proof is exactly like the corresponding proof for Σ_0–liftups ((1) in Lemma 2.7.5). In particular we have: $\pi^* : M \rightarrow_{\Sigma_0} \mathbb{D}$. Repeating the proof of (2) in Lemma 2.7.5 we get:

(2) $\mathbb{D} \models$ Extensionality.
Hence $\cong$ is again a congruence relation and we can factor $\mathbb{D}$, getting:

$$\hat{\mathbb{D}} = (\mathbb{D} \backslash \cong) = \langle \hat{D}, \hat{\in}, \hat{A}, \hat{B}\rangle$$

where

$$\hat{D} =: \{\hat{s} | s \in D\},\ \hat{s} =: \{t | t \cong s\} \text{ for } s \in D$$
$$\hat{s} \hat{\in} \hat{t} \leftrightarrow: s \tilde{\in} t$$
$$\hat{A}\hat{s} \leftrightarrow: \tilde{A}s,\ \hat{B}\hat{s} \leftrightarrow: \tilde{B}s$$

Then $\hat{\mathbb{D}}$ is a well founded identity model satisfying extensionality. By Mostowski's isomorphism theorem there is an isomorphism k of $\hat{\mathbb{D}}$ onto M', where $M' = \langle |M'|, \in, A', B'\rangle$ is transitive. Set:

$$[s] =: k(\hat{s}) \text{ for } s \in D$$
$$\pi'(x) =: [x^*] \text{ for } x \in M$$
$$H_i =: \{\hat{s} | s \in D_i\} (i < n \text{ or } i = n < \omega).$$

We shall *initially* interpret the variables v^i on the M' side as ranging over H_i. We call this the *pseudo interpretation.* Later we shall show that it (almost) coincides with the intended interpretation. By (1) we have

(3) $\pi' : M \rightarrow_{\Sigma_0^{(n)}} M'$ in the pseudo interpretation. (Hence $\pi' : M \rightarrow_{\Sigma_0^{(n)}} M'$.)

Lemma 2.7.19 then follows from:

Lemma 2.7.20. *$\langle M', \pi'\rangle$ is the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$.*

For $n = 0$ this was proven in Lemma 2.7.6, so assume $n > 0$. We again use the abbreviation:

$$[f, x] =: [\langle f, x\rangle] \text{ for } \langle f, x\rangle \in D.$$

Defining $\tilde{H}$ exactly as in the proof of Lemma 2.7.6, we can literally repeat our previous proofs to get:

(4) $\tilde{H}$ is transitive.

(5) $[f, x] = \pi(f)(x)$ if $f \in H$ and $\langle f, x\rangle \in D$. (Hence $\tilde{H} = H'$.)

(6) $\pi' \supset \pi$.

(However (7) in Lemma 2.7.6 will have to be proven later.)

In order to see that $\pi : M \to_{\Sigma^{(n)}} M'$ in the intended interpretation we must show that $H_i = H^i_M$, for $i < n$ and that $H_n \subset H^n_M$. As a first step we show:

(7) H_i is transitive for $i \leq n$.

Proof: Let $s \in H_i, t \in s$. Let $s = [f, x]$ where $f \in \Gamma^n_i$. We must show that $t = [g, y]$ for $g \in \Gamma^n_i$. Let $t = [g', y]$. Then $\langle y, x\rangle \in \pi(e)$ where

$$e = \{\langle u, v\rangle | g'(u) \in f(v)\}.$$

Set:

$$a =: \{u | g'(u) \in \mathrm{rng}(f)\}, g = g' \upharpoonright a.$$

Claim 1 $g \in \Gamma^n_i$.

Proof: $a \subset \mathrm{dom}(q')$ is $\underline{\Sigma}^{(n)}_0$. Hence $a \in H$ and $g \in \Gamma^n$. If $i < n$, then $\mathrm{rng}(g) \subset \mathrm{rng}(f) \subset H^i_M$. Hence $g \in \Gamma^n_i$. Now let $i = n$. Then $\mathrm{rng}(f) \in \Gamma^n_n$ and the relation $z = g(y)$ is $\underline{\Sigma}^{(n)}_0$. Hence $g \in H^n_M$.

QED (Claim 1)

Claim 2 $t = [g, y]$

Proof:

$$\bigwedge u, v(\langle u, v\rangle \in e \to \langle u, u\rangle \in e')$$

where $e' = \{\langle u, w\rangle | g(u) = g'(w)\}$. Hence the same Π_1 statement holds of $\pi(e), \pi(e')$ in H'. Hence $\langle y, y\rangle \in \pi(e')$. Hence $[g, y] = [g', y] = t$.

QED (7)

We can improve (3) to:

(8) Let $\Psi = \bigvee v^{i_1}_{v_1}, \ldots, v^{i_r}_r \varphi$, where φ is $\Sigma^{(n)}_0$ and $i_l < n$ or $i_l = n < \omega$ for $l = 1, \ldots, r$. Then π' is "Ψ–elementary" in the sense that:

$$M \models \Psi[\vec{x}] \leftrightarrow M' \models \Psi[\pi'(\vec{x})] \text{ in the pseudo interpretation.}$$

Proof: We first prove ($\rightarrow$). Let $M \models \varphi[\vec{z}, \vec{x}]$. Then $M' \models \varphi[\pi'(\vec{z}), \pi'(\vec{x})]$ by (3).

We now prove ($\leftarrow$). Let:

$$M' \models \varphi[[f_1, z_1], \ldots, [f_r, z_r], \pi'(\vec{x})]$$

where $f_l \in \Gamma^n_{i_l}$ for $l = 1, \ldots, r$. Since $\pi'(x) = [\text{const}_x, 0]$, we then have: $\langle z_1, \ldots, z_r, 0 \ldots 0\rangle \in \pi(e)$, where:

$$e = \{\langle u_1, \ldots, u_r, 0 \ldots 0\rangle : M \models \varphi[\vec{f}(\vec{u}), \vec{x}]\}.$$

Hence $e \neq \emptyset$. Hence

$$\bigvee v_1 \ldots v_r M \models \varphi[\vec{f}(\vec{v}), \vec{x}]$$

where $\text{rng}(f_l) \subset H^{i_l}$ for $l = 1, \ldots, r$. Hence $M \models \Psi[\vec{x}]$. QED (8)

If $i < n$, then every $\Pi^{(i)}_1$ formula is $\Sigma^{(n)}_0$. Hence by (8):

(9) If $i < n$ then

$$\pi' : M \rightarrow_{\Sigma^{(i)}_2} M' \text{ in the pseudo interpretation.}$$

We also get:

(10) Let $n < w$. Then:

$$\pi' \upharpoonright H^n_M : H^n_M \rightarrow_{\Sigma_0} H_n \text{ cofinally.}$$

Proof: Let $x \in H_n$. We must show that $x \in \pi'(a)$ for an $a \in H^n_M$. Let $x = [f, y]$, where $f \in \Gamma^n_n$. Let $d = \text{dom}(f), a = \text{rng}(f)$. Then $y \in \pi(d)$ and:

$$\bigwedge z \in d \ \langle z, 0\rangle \in e$$

where

$$\begin{aligned} e &= \{\langle u, v\rangle | f(u) \in \text{const}_a(v)\} \\ &= \{\langle u, 0\rangle | f(u) \in a\}. \end{aligned}$$

This is a Σ_0 statement about d, e. Hence the same statement holds of $\pi(d), \pi(e)$ in H_n. Hence $\langle z, 0\rangle \in \pi(e)$. Hence $[f, y] \in \pi'(a)$. QED (10)

(**Note:** (10) and (3) imply that $\pi' : M \rightarrow_{\Sigma^{(n)}_1} M'$ is the pseudo interpretation, but this also follows directly from (8).)

Letting $M = \langle J^A_\alpha, B\rangle$ and $M' = \langle |M'|, A', B'\rangle$ we define:

$$M_i = \langle H^i_M, A \cap H^i_M, B \cap H^i_M\rangle, M'_i = \langle H_i, A' \cap H_i, B' \cap H_i\rangle$$

for $i < n$ or $i = n < w$. Then each M_i is acceptable. It follows that:

(11) M'_i is acceptable.

Proof: If $i = n$, then $\pi' \upharpoonright M_n : M_n \to_{\Sigma_0} M'_n$ cofinally by (3) and (10). Hence M'_n is acceptable by §5 Lemma 2.5.5. If $i < n$, then $\pi' \upharpoonright M_i : M_i \to_{\Sigma_2^{(i)}} M'_i$ by (9). Hence M'_i is acceptable since acceptability is a Π_2 condition. QED (11)

We now examine the "correctness" of the pseudo interpretation. As a first step we show:

(12) Let $i + 1 \leq n$. Let $A \subset H_{i+1}$ be $\underset{\sim}{\Sigma}_1^{(i)}$ in the pseudo interpretation. Then $\langle H_{i+1}, A\rangle$ is amenable.

Proof: Suppose not. Then there is $A' \subset H_{i+1}$ such that A' is $\underset{\sim}{\Sigma}_1^{(i)}$ in the pseudo interpretation, but $\langle H_i, A'\rangle$ is not amenable. Let:

$$A'(x) \leftrightarrow B'(x, p)$$

where B' is $\Sigma_1^{(i)}$ in the pseudo interpretation. For $p \in M'$ we set:

$$A'_p =: \{x | B'(x, p)\}.$$

Let B be $\Sigma_1^{(i)}(M)$ by the same definition. For $p \in M$ we set:

$$A_p =: \{x | B(x, p)\}.$$

Case 1 $i + 1 < n$.

Then $\bigvee p \bigvee a^{i+1} \bigwedge b^{i+1} b^{i+1} \neq a^{l+1} \cap A'_p$ holds in the pseudo interpretation. This has the form: $\bigvee p \bigvee a^{i+1} \varphi(p, a^{i+1})$ where φ is $\Pi_1^{(i+1)}$, hence $\Sigma_0^{(n)}$ in the pseudo interpretation. By (8) we conclude that $M \models \varphi(p, a^{i+1})$ for some $p, a^{i+1} \in M$. Hence $\langle H_M^{i+1}, A_p\rangle$ is not amenable, where A_p is $\underset{\sim}{\Sigma}_1^{(i)}(M)$.
Contradiction! QED (Case 1)

Case 2 Case 1 fails.

Then $i + 1 = n$. Since π' takes H_M^n cofinally to H_n. There must be $a \in H_M^n$ such that $\pi(a) \cap A' \notin H_n$. From this we derive a contradiction. Let $A' = A'_p$ where $p = [f, z]$. Set: $\tilde{B} = \{\langle z, w\rangle | B(w, f(z))\}$. Then $\tilde{B}$ is $\underset{\sim}{\Sigma}_1^{(i)}(M)$. Set: $b = (d \times a) \cap \tilde{B}$, where $d = \text{dom}(f)$. Then $b \in H_M^n$. Define $g : d \to H_M^n$ by:

$$g(z) =: A_{f(z)} \cap a = \{x \in a | \langle z, x\rangle \in b\}.$$

Then $g \in H_M^n$, since it is rudimentary in $a, b \in H_M^n$. Let $\varphi(u^n, v^n, w)$ be the $\Sigma_0^{(n)}$ statement expressing

$$u = A_w \cap v^n \text{ in } M.$$

Then setting:

$$e = \{\langle v, 0, w\rangle | M \models \varphi[g(v), a, f(z)]\}$$

we have:

$$\bigwedge v \in d \; \langle v, 0, v\rangle \in e.$$

But then the same holds of $\pi(d), \pi(e)$ in H_n. Hence $\langle z, 0, z\rangle \in \pi(e)$. Hence: $[g, z] = A_{[f,z]} \cap \pi(a) \in H_n$.
Contradiction! QED (12)

On the other hand we have:

(13) Let $i+1 < n$. Let $A \subset H_M^{i+1}$ be $\Sigma_1^{(i)}(M)$ in the parameter p such that $A \notin M$. Let A' be $\Sigma_1^{(i)}(M')$ in $\pi'(p)$ by the same $\Sigma_1^{(i)}(M')$ definition in the pseudo interpretation. Then $A' \cap H_{i+1} \notin M'$.

Proof: Suppose not. Then in M' we have:

$$\bigvee a \bigwedge v^{i+1}(v^{i+1} \in a \leftrightarrow A'(v^{i+1})).$$

This has the form $\bigvee a \varphi(a, \pi(p))$ where φ is $\Pi_1^{(i+1)}$ hence $\Sigma_0^{(n)}$. By (8) it then follows that $\bigvee a \varphi(a, p)$ holds in M. Hence $A \in M$.
Contradiction! QED (13)

Recall that for any acceptable $M = \langle J_\alpha^A, B\rangle$ we can define ρ_M^i, H_M^i by:

$$\begin{aligned}
\rho^0 = &\ \alpha \\
\rho^{i+1} = &\ \text{the least } \rho \text{ such that there is } A \text{ which is} \\
&\ \underline{\Sigma}_1^{(i)}(M) \text{ with } A \cap \rho \notin M \\
H^i = &\ J_{\rho_i}[A].
\end{aligned}$$

Hence by (11), (12), (13) we can prove by induction on i that:

(14) Let $i < n$. Then

(a) $\rho_{M'}^i = \rho_i,\ H_{M'}^i = H_i$

(b) The pseudo interpretation is correct for formulae φ, all of whose variables are of type $\leq i$.

By (9) we then have:

(15) $\pi' : M \to_{\Sigma_2^{(i)}} M'$ for $i < n$.
This means that if $n = \omega$, then π' is automatically Σ^*–preserving. If $n < \omega$, however, it is not necessarily the case that $H_n = H_M^n$, — i.e. the pseudo interpretation is not always correct. By (12), however we do have:

(16) $\rho_n \leq \rho^n_M$, (hence $H_n \subset H^n_{M'}$).
Using this we shall prove that π' is $\Sigma_0^{(n)}$–preserving. As a preliminary we show:

(17) Let $n < w$. Let φ be a $\Sigma_0^{(n)}$ formula containing only variables of type $i \leq n$. Let $v_1^{i_1}, \ldots, v_r^{i_r}$ be a good sequence for φ. Let $x_1, \ldots, x_r \in M'$ such that $x_l \in H_{i_l}$ for $l = 1, \ldots, r$. Then $M \models \varphi[x_1, \ldots, x_r]$ holds in the correct sense iff it holds in the pseudo interpretation.

Proof: (sketch)
Let C_0 be the set of all such φ with: φ is $\Sigma_1^{(i)}$ for an $i < n$. Let C be the closure of C_0 under sentential operation and bounded quantifications of the form $\bigwedge v^n \in w^n \varphi$, $\bigvee v^n \in w^n \varphi$. The claim holds for $\varphi \in C_0$ by (15). We then show by induction on φ that it holds for $\varphi \in C$. In passing from φ to $\bigwedge v^n \in w^n \varphi$ we use the fact that w^n is interpreted by an element of H_n. QED (17)

Since $\pi''' H^i_M \subset H_i$ for $i \leq n$, we then conclude:

(18) $\pi' : M \to_{\Sigma_0^{(n)}} M'$.
It now remains only to show:

(19) $[f, x] = \pi'(f)(x)$.

Proof: Let $f(x) = G(x, p)$ for $x \in \text{dom}(f)$, where G is $\Sigma_1^{(j)}$ good for a $j < n$. Let $a = \text{dom}(f)$. Let $\Psi(u, v, w)$ be a good $\Sigma_1^{(j)}$ definition of G. Set:

$$e = \{\langle z, y, w\rangle | M \models \Psi[f(z), \text{id}_a(y), \text{const}_p(w)]\}.$$

Then $z \in a \to \langle z, z, 0\rangle \in e$. Hence the same holds of $\pi(a), \pi(e)$. But $x \in \pi(a)$. Hence:

$$M' \models \Psi[[f, x], [\text{id}_a, x], [\text{const}_p, x]],$$

where $[\text{id}_a, x] = x, [\text{const}_p, 0] = \pi'(p)$. Hence:

$$[f, x] = G'(x, \pi'(p)) = \pi'(f)(x),$$

where G' has the same $\Sigma_1^{(j)}$ definition. QED (19)

Lemma 2.7.20 is then immediate from (6), (18) and (19).
QED (Lemma 2.7.19)

As a corollary of the proof we have:

Lemma 2.7.21. *Let $\langle M', \pi'\rangle$ be the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$. Let $i < n$. Then* *(a)* $\pi' : M \to_{\Sigma_2^{(i)}} M'$

(b) *If* $\rho_M^i \in M$*, then* $\pi'(\rho_M^i) = \rho_M^i$.

(c) *If* $\rho_M^i = \mathrm{On}_M$*, then* $\rho_{M'}^i = \mathrm{On}_{M'}$.

Proof:

(a) follows by (9) and (14).

(b) In M we have:

$$\bigwedge \xi^0 \bigvee \xi^i (\xi^0 < \rho_M^i \leftrightarrow \xi^0 = \xi^i).$$

This has the form $\bigwedge \xi^0 \Psi(\xi^0, \rho_M^i)$ where Ψ is $\Sigma_0^{(n)}$. But then the same holds of $\pi'(\rho_M^i)$ in M' by (8) and (14) — i.e.

$$\bigwedge \xi^0 \bigvee \xi^i (\xi^0 < \pi(\rho_M^i) \leftrightarrow \xi^0 = \xi^i).$$

(c) In M we have $\bigwedge \xi^0 \bigvee \xi^i \xi^0 = \xi^i$, hence the same holds in M' just as above.

QED (Lemma 2.7.21)

The *interpolation lemma* for $\Sigma_0^{(n)}$ liftups reads:

Lemma 2.7.22. *Let* $\sigma : H' \to_{\Sigma_0} |M^*|$ *and* $\pi^* : M \to_{\Sigma_0^{(n)}} M^*$ *such that* $\pi^* \supset \sigma\pi$*. Then the* $\Sigma_0^{(n)}$ *liftup* $\langle M', \pi' \rangle$ *of* $\langle M, \pi \rangle$ *exists. Moreover there is a unique map* $\sigma' : M' \to_{\Sigma_0^{(n)}} M^*$ *such that* $\sigma' \restriction H' = \sigma$ *and* $\sigma'\pi' = \pi^*$.

Proof: $\tilde{\in}$ is well founded since:

$$\langle f, x \rangle \tilde{\in} \langle g, y \rangle \leftrightarrow \pi^*(f)(\sigma(x)) \in \pi^*(g)(\sigma(y)).$$

Thus $\langle M', \pi' \rangle$ exists. But for $\Sigma_0^{(n)}$ formulae $\varphi = \varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ we have:

$$\begin{aligned}
&M' \models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)v_r)] \\
&\leftrightarrow \langle x_1, \ldots, x_n \rangle \in \pi(e) \\
&\leftrightarrow \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in \sigma(\pi(e)) = \pi^*(e) \\
&\leftrightarrow M^* \models \varphi[\pi^*(f_1)(\sigma(x_1)), \ldots, \pi^*(f_r)(\sigma(x_r))]
\end{aligned}$$

where:

$$e = \{\langle x_1, \ldots, x_r \rangle | M \models \varphi[f_1(x_1), \ldots, f_r(x_r)]\}$$

and $\langle f_l, x_l \rangle \in \Gamma^n_{i_l}$ for $i = 1, \ldots, r$. Hence there is a $\Sigma_0^{(n)}$–preserving embedding $\sigma : M' \to M^*$ defined by:

$$\sigma'(\pi'(f)(x)) = \pi^*(f)(\sigma(x)) \text{ for } \langle f, x \rangle \in \Gamma^n.$$

Clearly $\sigma' \restriction H' = \sigma$ and $\sigma'\pi' = \pi^*$. But σ' is the unique such embedding, since if $\tilde{\sigma}$ were another one, we have

$$\tilde{\sigma}(\pi'(f)(x)) = \pi^*(f)(\sigma(x)) = \sigma'(\pi'(f)(x)).$$

QED (Lemma 2.7.22)

We can improve this result by making stronger assumptions on the map π, for instance:

Lemma 2.7.23. *Let $\langle M^*, \pi^* \rangle$ be the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi \rangle$. Let $\pi^* \restriction \rho_M^{n+1} = \mathrm{id}$ and $\mathbb{P}(\rho_M^{n+1}) \cap M^* \subset M$. Then $\rho^n_{M^*} = \sup \pi^{*''} \rho^n_M$.*

(Hence the pseudo interpretation is correct and π^* is $\Sigma_1^{(n)}$ preserving.)

Proof: Suppose not. Let $\tilde{\rho} = \sup \pi^{*''} \rho^n_M < \rho^n_{M^*}$. Set:

$$H^n = H^n_M = J^{A_M}_{\rho^n_M};\ \tilde{H} = J^{A_M}_{\tilde{\rho}}.$$

Then $\tilde{H} \in M^*$. Let A be $\Sigma_1^{(n)}(M)$ in p such that $A \cap \rho_M^{n+1} \notin M$. Let:

$$Ax \leftrightarrow \bigvee y^n B(y^n, x),$$

where B is $\Sigma_0^{(n)}$ in p. Let B^* be $\Sigma_0^{(n)}(M^*)$ in $\pi^*(p)$ by the same definition. Then

$$\pi^* \restriction H^n : \langle H^n, B \cap H^n \rangle \to_{\Sigma_1} \langle \tilde{H}, B^* \cap \tilde{H} \rangle.$$

Then $A \cap \rho_M^{n+1} = \tilde{A} \cap \rho_M^{n+1}$, where:

$$\tilde{A} = \{x | \bigvee y^n \in \tilde{H}\ B^*(y, x)\}.$$

But $\tilde{A}$ is $\Sigma_1^{(n)}(M^*)$ in $\pi^*(p)$ and $\tilde{H}$. Hence

$$A \cap \rho_M^{n+1} = \tilde{A} \cap \rho_M^{n+1} \in \mathbb{P}(\rho_M^{n+1}) \cap M^* \subset M.$$

Contradiction! QED (Lemma 2.7.23)