Chapter 3

Mice

3.1 Introduction

In this chapter we develop some of the tools needed to construct fine structural inner models which go beyond L. The concept of "mouse" is central to this endeavor. We begin with a historical introduction which traces the genesis of that notion. This history, and the concepts which it involves, are familiar to many students of set theory, but the thread may grow fainter as the history proceeds. If you, the present reader, find the introduction confusing, we advise you to skim over it lightly and proceed to the formal development in §3.2. The introduction should then make more sense later on.

Fine structure theory was originally developed as a tool for understanding the constructible hierarchy. It was used for instance in showing that $V = L$ implies $\square_\beta$ for all infinite cardinals β, and that every non weakly compact regular cardinal carries a Souslin tree. It was then used to prove the covering lemma for L, a result which pointed in a different direction. It says that, if there is no non trivial elementary embedding of L into itself, then every uncountable set of ordinals is contained in a constructible set having the same cardinality. This implies that if any $\alpha \geq \omega_2$ is regular in L, then its cofinality is the same as its cardinality. In particular, successors of singular cardinals are absolute in L. Any cardinal $\alpha \geq \omega_2$ which is regular in L remains regular in V. In general, the covering lemma says that despite possible local irregularities and cofinalities in L is retained in V.

If, however, L can be imbedded non trivially into itself, then the structure of cardinalities and cofinalities in L is virtually wiped out in V. There is then a countable object known as $0^{\#}$ which encodes complete information about the class L and a non trivial embedding of L. $0^{\#}$ has many concrete representations, one of the most common being a structure $L^U_\nu = \langle L_\nu[U], \in, U\rangle$, where ν is the successor of an inaccessible cardinal κ in L and U is a normal ultrafilter on $\mathbb{P}(\kappa) \cap L$. (Later, however, we shall find it more convenient to work with extenders than with ultrafilters.) This structure, call it M_0, is *iterable* , giving rise to iterates $M_i (i < \infty)$ and embedding $\pi_{ij} : M_i \to_{\Sigma_0} M_j$ $(i \leq j < \infty)$. The iteration points κ_i $(i < \infty)$ are called the *indiscernables* for L and form a closed proper class of ordinals. Each κ_c is inaccessible in L. Thus there are unboundedly many inaccessibles of L which become ω–cofinal cardinals in V. It can also be shown that all infinite successor cardinals in L are collapsed and become ω–cofinal in V. If we chose κ_0 minimally, then $M_0 = 0^{\#}$ is unique. We briefly sketch the argument for this, since it involves a principle which will be of great importance later on. By the minimal choice of κ_0 it can be shown that $h_{M_0}(\emptyset) = M_0$ (i.e. $\rho^1_{M_0} = \omega$ and $\emptyset \in P^1_{M_0}$). Now let $M'_0 = L^{U'_0}_{\nu'_0}$ be another such structure. Iterate M_0, M'_0 out to ω_1, getting iteration $\langle M_i | i \leq \omega_1\rangle$, $\langle M'_i | i \leq \omega_1\rangle$ with iteration points κ_i, κ'_i. Then $\kappa_{\omega_1} = \kappa'_{\omega_1} = \omega_1$. Moreover the sets:

$$C = \{\kappa_i | i < \omega_1\}, C' = \{\kappa'_i | i < \omega_1\}$$

are club in ω_1. Hence $C \cap C'$ is club in ω_1. But the ultrafilters $U_{\omega_1}, U'_{\omega_1}$ are uniquely determined by $C \cap C'$. Hence $M_{\omega_1} = M'_{\omega_1}$. But then:

$$M_0 \simeq h_{M_{\omega_1}}(\emptyset) = h_{M'_{\omega_1}}(\emptyset) \simeq M'_0.$$

Hence $M_0 = M'_0$. This *comparision iteration* of two iterable structures will play a huge role in later chapters of this book.

The first application of fine structure theory to an inner model which significantly differed from L was made by Solovay in the early 1970's. Solovay developed this fine structure of L^U (where U is a normal measure on $\mathbb{P}(\kappa) \cap L^U$). He showed that each level $M = J^U_\alpha$ had a viable fine structure, with $\rho^n_M, P^n_M, R^n_M (n < \omega)$ defined in the usual way, although M might be neither acceptable nor sound. If e.g. $\alpha > \kappa$ and $\rho^1_M < \kappa$ (a case which certainly occurs), the we clearly have $R^1_M = \emptyset$. However, M has a standard parameter $p = p_M \in P^1_M$ and if we transitivize $h_M(P)$, we get a structure $\overline{M} = J^{\overline{U}}_{\overline{\alpha}}$ which iterates up to M in κ many steps. $\overline{M}$ is then called the *core* of M. ($\overline{M}$ itself might still not be acceptable, since a proper initial segment of $\overline{M}$ might not be sound.) (If $n < 1$ and $\rho^n_M < \kappa$, we can do essentially the same analysis, but when iterating $\overline{M}$ to M we must use $\Sigma^{(n)}_0$–preserving ultrapowers, as defined in the next section.)

Dodd and Jensen then turned Solovay's analysis on its head by defining a *mouse (or Solovay mouse)* to be (roughly) any J_α or iterable structure of the form $M = J^U_\alpha$ where U is a normal measure at some κ on M and $\rho^\omega_M \leq \kappa$. They then defined the *core model* K to be the union of all Solvay mice. They showed that, if there is no non trivial elementary embedding of K into K, then the covering lemma for K holds. If, on the other hand, there is such an embedding π with critical point κ, then U is a normal measure on κ in $L^U = \langle L[u], \in, u\rangle$, where:

$$U = \{x \in \mathbb{P}(\kappa) \cap K | \kappa \in \pi(X)\},$$

(This showed, in contrast to the prevailing ideology, that an inner model with a measurable cardinal can indeed be "reached from below".) The simplest Solovay mouse is $0^\#$ as described above. What K is depends on what there is. If $0^\#$ does not exist, then $K = L$. If $0^\#$ exists but $0^{\#\#}$ does not, then $K = L(0^\#)$ etc. In order to define the general notion of Solovay mouse, one must employ the full paraphanalia of fine stucture theory.

Thus we have reached the situation that fine structure theory is needed not only to analyze a previously defined inner model, but to define the model itself.

If we have reached L^U with U a normal ultrafilter on κ and $\tau = \kappa^+$ in L^U, then we can regard L^U_τ as the "next mouse" and continue the process. If $(L^\kappa_\tau)^\#$ does not exist, however, this will mean that L^U is the core model. The full covering lemma will then not necessarily hold, since V could contain a Prikry sequence for κ.

However, we still get the *weak covering lemma*:

$$cf(\beta) = \text{card}(\beta) \text{ if } \beta \geq \omega_2 \text{ is a cardinal in } K.$$

We also have *generic absoluteness*:

The definition of K is absolute
in every set generic extension of V.

In the ensuring period a host of "core model constructions" were discovered. For instance the "core model below two measurables" defined a unique model with the above properties under the assumption that there is no inner model with two measurable cardinals. Similarly with the "core model up to a measurable limit of measurables" etc. Initially this work was pursued by Dodd and Jensen, on the one hand, and by Bill Mitchell on the other. Mitchell got further, introducing several important innovations. He divided the construction of K into two stages: In the first he constructed an inner model K^C, which may lack the two properties stated above. He then "extracted" K from K^C, in the process defining an elementary embedding of K into K^C. This approach has been basic to everything done since. Mitchell also introduced the concept of *extenders*, having realized that the normal ultrafilters alone could not code the embeddings involved in constructing K.

There are many possible concrete representations of mice, but in general a mouse is regarded as a structure $M = J_\nu^E$ where E describes an indexed sequence of ultrafilters or extenders. A major requirement is that M be *iterable* , which entails that any of the indexed extenders or ultrafilters can be employed in the iteration. But this would seem to imply that eny F lying on the indeved sequence must be *total* — i.e. an ultrafilter or extender on the whole of $\mathbb{P}(\kappa) \cap M$ (κ being the critical point). Unfortunately the most natural representations of mice involve "allowing extenders (or ultrafilters) to die". Letting $M = J_\nu^U$ be the representation of $0^\#$ described above, it is known that $\rho_M^1 = \omega$. Hence $J_{\nu+1}^U$ contains new subsets of κ which are not "measured" by the ultrafilter U. The natural representation of $0^{\#\#}$ would be $M' = J_{\nu'}^{U,U'}$ where:

$$U' = \{X | \kappa' \in \pi(x)\},$$

and π is an embedding of L^U into itself with critical point $\kappa' > \kappa$. But U is not total. How can one iterate such a structure? Because of this conundrum, researchers for many years followed Solovay's lead in allowing only total ultrafilters and extenders to be indexed in a mouse. Thus Solovay's representation of $0^{\#\#}$ was $J_{\nu'}^{U'}$ This structre is not acceptable, however, since there is a $\gamma < \nu'$ set. $\kappa' < \gamma$ and $\rho_{J_\gamma^U}^1 = \omega < \kappa'$. Such representation of mice were unnatural and unwieldy. The conundrum was finally resolved by Mitchell and Stewart Baldwin, who observed that the structures in which extenders are "allowed to die" are in fact, iterable in a very good sense. We shall deal with this in §3.4. All of the innovations mentioned here were then incorporated into [MS] and [CMI]. They where also employed in [MS] and [NFS].

It was originally hoped that one could define the core model below virtually any large cardinal — i.e. on the assumption that no inner model with the cardinal exists one could define a unique inner model K satisfying weak covering and generic absoluteness. It was then noticed, however, that if we assume the existence of a Woodin cardinal, then the existence of a definable K with the above properties is provably false. (This is because Woodin's "stationary tower" forcing would enable us to change the successor of ω_ω while retaining ω_ω as a singular cardinal. Hence, by the covering lemma, K would have to change.) This precludes e.g. the existence of a core model below "an inaccessible above a Woodin", but it does not preclude constructing a core model below one Woodin cardinal. That is, in fact, the main theorem of this book: Assuming that no inner model with a Woodin cardinal exists, we define K with the above two properties.

In 1990 John Steel made an enormous stride toward achieving this goal by proving the following theorem: Let κ be a measurable cardinal. Assume that V_κ has no inner model with a Woodin cardinal. Then there is V–definable inner model K of V_κ which, relativized to V_κ, has he above two properties. This result, which was exposited in [CMI] was an enormous breakthrough, which laid the foundation for all that has been done in inner model theory since then. There remained, however, the pesky problem of doing without the measurable — i.e. constructing K and proving its properties assuming only "ZFC+ there is no inner model with a Woodin". The first step was to construct the model K^C from this assumption. This was almost achieved by Mitchell and Schindler in 2001, except that they needed the additional hypothesis: GCH. Steel then showed that this hypothesis was superfluous. These results were obtained by directly weakening the "background condition" originally used by Steel in constructing K^C. The result of Mitchell and Schindler were published in [UEM]. Independently, Jensen found a construction of K^C using a different background condition called "robustness". This is exposited in [RE]. There reamained the problem of extracting a core model K from K^C. Jensen and Steel finally achieved this result in 2007. It was exposited in [JS].

In the next section we deal with the notion of *extenders*, which is essential to the rest of the book. (We shall, however, deal only with so called "short extenders", whose length is less than or equal to the image of the critical point.)

3.2 Extenders

The *extender* is a generalization of the normal ultrafilter. A normal ultrafilter at κ can be described by a two valued function on $\mathbb{P}(\kappa)$. An extender, on the other hand, is characterized by a map of $\mathbb{P}(\kappa)$ to $\mathbb{P}(\lambda)$, where $\lambda > \kappa$. λ is then called the *length* of the extender. Like a normal ultrafilter an extender F induces a canonical elementary embedding of the universe V into an inner model W. We express this in symbols by: $\pi : V \to_F W$. W is then called the *ultrapower* of V by F and π is called the *canonical embedding* induced by F. The pair $\langle W, \pi\rangle$ is called the *extension* of V by F. We will always have: $\lambda \leq \pi(\kappa)$. However, just as with ultrafilters, we shall also want to apply extenders to transitive models M which may be smaller than V. F might then not be an element of M. Moreover $\mathbb{P}(\kappa)$ might not be a subset of M, in which case F is defined on the smaller set $U = \mathbb{P}(\kappa) \cap M$. Thus we must generalize the notion of extenders, countenancing "suitable" subsets of $\mathbb{P}(\kappa)$ as extender domains. (However, the ultrapower of M by F may not exist.)