QED(Lemma 3.2.28)

3.3 Premice

A major focus of modern set theory is the subject of "strong axioms of infinity". These are principles which posit the existence of a large set or class, not provable in ZFC. Among these principles are the *embedding axioms*, which posit the existence of a non trivial elementary embedding of one inner model into another. The best known example of this is the *measurability axiom*, which posits the existence of a non trivial elementary embedding π of V into an inner model. ("Non trivial" here means simply that $\pi \neq \text{id}$. Hence there is a unique *critical point* $\kappa = \text{crit}(\pi)$ such that $\pi \upharpoonright \kappa = \text{id}$ and $\pi(\kappa) > \kappa$.) The critical point κ of π is then called a *measurable cardinal*, since the existence of such an embedding is equivalent to the existence of an ultrafilter (or *two valued measure*) on κ.

This is a typical example of the recursing case that an axiom positing the existence of a proper class (hence not formulable in ZFC) reduces to a statement about set existence. The weakest embedding axiom posits the existence of a non trivial embedding of L into itself. This is equivalent to the existence of a countable transitive set called $0^{\#}$, which can be coded by a real number. (There are many representations of $0^{\#}$, but all have the same degree of constructability.) The "small" object $0^{\#}$ in fact contains complete information about both the proper class L and an embedding of L into itself. We can then form $L(0^{\#})$, the smallest universe containing the set $0^{\#}$. If $L(0^{\#})$ is embeddable into itself we get $0^{\#\#}$, which gives complete information about $L(0^{\#})$ and its embedding ... etc. This process can be continued very far. Each stage in this progression of embeddings, leading to larger and larger universes, is coded by a specific set, called a *mouse*. $0^{\#}$ and $0^{\#\#}$ are the first two examples of mice. It is not yet known how far this process goes, but it is conjectured that all stages can be represented by mice, as long as the embeddings are representable by extenders. (Extenders in our sense are also called *short extenders*, since one must modify the notion in order to go still further.) The concept of mouse, however hard it is to explicate, will play a central role in this book.

We begin, therefore, with an informal discussion of the *sharp operation* which takes a set a to $a^{\#}$, since applications of this operation give us the smallest mice $0^{\#}, 0^{\#\#}$, etc.

Let a be a set such that $a \in L[a]$. Suppose moreover that there is an elementary embedding π of $L^a = \langle L[a], \in, a\rangle$ into itself such that $a \in L^a_\kappa$, where $\kappa = \text{crit}(\pi)$. We also assume without loss of generality, that κ is minimal for π with this property. Let $\tau = \kappa^{+L^a}$ and $\nu = \sup \pi''\tau$. Then $\tilde{\pi} : L^a_\tau \prec L^a_\nu$ cofinally, where $\tilde{\pi} = \pi \upharpoonright L^a_\tau$. Set $F = \pi \upharpoonright \mathbb{P}(\kappa)$. F is then an extender at κ with base $L_\tau[a]$ and extension $\langle L_\nu[a], \tilde{\pi}\rangle$.

$\langle L^a_\nu, F\rangle = \langle L_\nu[a], a, F\rangle$ is then amenable by Lemma 3.2.2. It can be shown, moreover, that F is uniquely defined by the above condition. We then define:

Definition 3.3.1. $a^\#$ is the structure $\langle L_\nu[a], a, F\rangle$.

Note. In the literature $a^\#$ has many different representations, all of which have the same constructibility degree as $\langle L_\nu[a], a, F\rangle$.

$a^\#$ has a number of interesting properties, which we state here without proof. F is clearly an extender at κ on $\langle L^a_\nu, F\rangle$. Moreover, we can form the extension:

$$\pi_0 : \langle L^a_\nu, F\rangle \to_F \langle L^a_{\nu_1}, F_1\rangle.$$

We then have $\pi_0 \supset \tilde{\pi}$, $\pi_0(\kappa) = \nu$. (In fact $\pi_0 = \pi' \upharpoonright L^a_\nu$.) But we can then apply F_1 to $\langle L^a_{\nu_1}, F_1\rangle \ldots$ etc. This can be repeated indefinitely, showing that $a^\#$ is *iterable* in the following sense:

There are sequences $\kappa_i, \tau_i, \nu_i, F_i (i < \infty)$ and $\pi_{ij} (i \leq j < \infty)$ such that

- $\kappa_0 = \kappa, \tau_0 = \tau, \nu_0 = \nu, F_0 = F$.
- $\kappa_{i+1} = \pi'_{i,i+1}(\kappa_i), \nu_i = \pi'_{i,i+1}(\pi_i), \tau_i = \kappa_i^{+L^a_{\nu_i}}$.
- F_i is a full extender at κ_i with base $L_{\tau_i}[a]$ and extension $\langle L_{\nu_i}[a], \pi'_{i,i+1} \upharpoonright L^{[a]}_{\tau_i}\rangle$.
- $\pi'_{i,i+1} : \langle L^a_{\nu_i}, F_i\rangle \to_{F_i} \langle L^a_{\nu_{i+1}}, F_{i+1}\rangle$.
- The maps π'_{ij} commute — i.e.
  $$\pi'_{ii} = \text{id};\ \pi'_{ij}\pi'_{hi} = \pi'_{hj}.$$
- For limit λ, $\langle L^a_{\nu_\lambda}, F_\lambda\rangle$, $\langle \pi'_{i\lambda} | i < \lambda\rangle$ is the transitivized direct limit of
  $$\langle\langle L^a_{\nu_0}, F_i\rangle | i < \lambda\rangle, \langle \pi'_{ij} | i \leq j < \lambda\rangle.$$

It turns out that $a^\# = \langle L^a_\nu, F\rangle$ is uniquely defined by the conditions:

- $\langle L^a_\nu, F\rangle$ is iterable in the above sense
- ν is minimal for such $\langle L^a_\nu, F\rangle$.

If $a = \emptyset$ we write: $0^\#$. $0^\# = \langle L_\nu, F\rangle$ is then acceptable. By a Löwenheim–Skolem type argument it follows that $0^\#$ is sound and $\rho^1_{0^\#} = \omega$. (To see this let $M = 0^\#, X = h_M(\omega)$. Let $\sigma : \overline{M} \stackrel{\sim}{\leftrightarrow} X$ be the transitivization of X, where $\overline{M} = \langle L_{\overline{\nu}}, \overline{F}\rangle$. Using the fact that $\sigma : \overline{M} \to M$ is Σ_1–preserving and M is iterable, it can be shown that $\overline{M}$ is iterable. Hence $\overline{M} = M$, since $\overline{\nu} \leq \nu$ and ν is minimal.) But then $0^\#$ is countable and can be coded by a real number. But this is real giving complete information about the proper class L, since we can recover the satisfaction relation for L by:

$$L \models \varphi[\vec{x}] \leftrightarrow L_{\kappa_i} \models \varphi[\vec{x}]$$

where i is chosen large enough that $x_1, \ldots, x_n \in L_{\kappa_i}$. But from $0^\#$ we also recover a nontrivial elementary embedding of L into itself, namely:

$$\pi : L \to_F L \text{ where } 0^\# = \langle L_\nu, F\rangle.$$

$0^\#$ is our first example of a mouse. All of its iterates, however, are not sound, since if $i > 0$, then $\mathrm{rng}(\pi_{0i}) = h_{M_i}(\omega)$, where $\rho^1_{M_i} = \rho^1_{M_0} = \omega$. But $\kappa_0 \notin \mathrm{rng}(\pi_{0i})$.

We can iterate the operation $\#$, getting $0, 0^\#, (0^\#)^\#, \ldots$ etc. This notation is not literally correct, however, since $a^\#$ is defined only when $a \in L[a]$. Thus, setting:

$$0^{\#(n)} = 0^{\overbrace{\# \cdots \#}^{n}},$$

we need to set: $0^{\#(n+1)} = (e^n)^\#$, where e^n codes $0, \ldots, 0^{\#(n)}$. If we do this in a uniform way, we can in fact define $0^{\#(\xi)}$ for all $\xi < \infty$.

Definition 3.3.2. Define $e^i, \nu_i, 0^{\#(i)} = \langle L^{e^i}_{\nu_i}, E_{\nu_i}\rangle (i < \infty)$ as follows:

$$e^i =: \{\langle x, \nu_i\rangle | j < i \wedge x \in E_{\nu_j}\} \text{ (hence } e^0 = \emptyset)$$
$$0^{\#(0)} =: \langle \emptyset, \emptyset\rangle \text{ (hence } \nu_0 = 0)$$
$$0^{\#(i+1)} =: (e^i)^\# \text{ (hence } \nu_{i+1} > \nu_i)$$

For limit λ we set:

$$\nu =: \sup_{i<\lambda} \nu_i,\ 0^{\#(\lambda)} =: \langle L^{e^\lambda}_{\nu_\lambda}, \emptyset\rangle,\ (\text{hence } \emptyset = E_{\nu_\lambda}).$$

By induction on $i < \infty$ it can be shown that each $0^{\#(i)}$ is acceptable and sound, although we skip the details here. Each $0^{\#(i)}$ is also iterable in a sense which we have yet to explicate. As before, it will turn out that the iterates are acceptable but not necessarily sound. Set:

$$E =: \bigcup_{i<\infty} e^i.$$

Then $L[E]$ is the smallest inner model which is closed under the $\#$ operation. (For this reason it is also called $L^{\#}$.) We of course set: $L^E =: \langle L[E], \in, E\rangle$.

L^E is a very L–like model, so much so in fact, that we can obtain the next mouse after all the $0^{\#(i)}(i < \infty)$ by repeating the construction of $0^{\#}$ with L^E in place of L: Suppose that $\pi : L^E \prec L^E$ is a nontrivial elementary embedding. Without loss of generality assume the critical point κ of π to be minimal for all such π. Let $\tau = \kappa^{+L^E}$ and $\nu = \sup \pi''\tau$. Then $\tilde{\pi} = \pi \restriction L^E_\tau$. Set: $F = \pi \restriction \mathbb{P}(\kappa)$. Then F is an extender with base $L_\tau[E]$ and extension $\langle L_\nu[E], \tilde{\pi}\rangle$. The new mouse is then $\langle L^E_\nu, F\rangle$.

As before, we can recover full information about L^E from $\langle L^E_\nu, F\rangle$ and we can recover a nontrivial embedding of L^E by: $\pi : L^E \to_F L^E$. $e = E \cup \{\langle x, \nu\rangle | x \in F\}$ then codes all the mice up to and including $\langle L^E_\nu, F\rangle$, so the next mouse is $e^{\#} \ldots$ etc.

Note. that $L^E||\nu = \langle L^E_\nu, \emptyset\rangle$ since, if $\kappa_i = \text{crit}(E_{\nu_{i+1}})$, then the sequence $\langle \kappa_i | i < \infty\rangle$ of all critical points of previous mice is discrete, whereas $\kappa = \text{crit}(F)$ is a fixed point of this sequence.

This process can be continued indefinitely. At each stage it yields a set which encodes full information about an inner model. We call these sets *mice*. Each mouse will be an acceptable structure of the form $M = \langle J^E_\alpha, E_\alpha\rangle$ where $E = \{\langle x, \nu\rangle | \nu < \alpha \wedge x \in E_\nu\}$ codes the set of 'previous' mice. For $\nu = \alpha$ we have: Either $E_\nu = \emptyset$ or ν is a limit ordinal and E_ν is a full extender at a $\kappa < \nu$ with extension $\langle J_\nu[E], \pi\rangle$ and base $J_\tau[E]$, where $\tau = \kappa^{+M}$.

For limit $\xi \leq \alpha$ we set: $M||\xi =: \langle J^E_\xi, E_\xi\rangle$. A class model L^E is called a *weasel* iff $E = \{\langle x, \nu\rangle | \nu < \infty \wedge x \in E_\nu\}$ and $L^E||\alpha =: \langle J^E_\alpha, E_\alpha\rangle$ is a mouse of all limit α.

When dealing with such structures M satisfying, we shall often use the following notation: If $E_\nu \neq \emptyset$, then $\kappa_\nu =$ the critical point of $E_\nu, \tau_\nu = \kappa^+ J^E_\nu$, and $\lambda_\nu =$ the length of $E_\nu = \pi(\kappa_\nu)$, where $\langle J^E_\nu, \pi\rangle$ is the extension of $J^E_{\tau_\nu}$ by E_ν.

In the above examples, the extenders E_ν were so small that τ_ν eventually got collapsed in $L[E_\nu]$. Thus E_ν was no longer an extender in $L[E_\nu]$, since it was not defined on all subsets of κ. However, if we push the construction far enough, we will eventually reach an E_ν which does not have this defect. $L[E_\nu]$ will then be the smallest inner model with a measurable cardinal.

In the above examples the extender E_ν is always generated by $\{\kappa_\nu\}$ Hence we could just as wel have worked with ultrafilters as with extenders. Eventually, however, we shall reach a point where genuine extenders are needed. In the examples we also chose $\lambda_\nu = \pi(\kappa_\nu)$ minimally — i.e. we imposed an *initial segment condition* which says that $E_\nu|\lambda$ is not a full extender for any $\lambda < \lambda_\nu$. This condition can become unduly restrictive, however: It might happen that we wish to add a new extender E_ν and that $E_\nu|\lambda$ is an extender which we added at an earlier stage. In that case we will have: $E_\nu|\lambda \in J^E_\nu$. In order to allow for this situation we modify the initial segment condition to read:

Definition 3.3.3. Let F be a full extender at κ with base S and extension $\langle S', \pi\rangle$. F satisfies the *initial segment condition* iff whenever $\lambda < \pi(\kappa)$ such that $F|\lambda$ is a full extender, then $F|\lambda \in S'$.

As indicated above, we expect our mice to be *iterable*. The example of an iteration given above is quite straightforward, but the general notion of iterability which we shall use is quite complex. We shall, therefore, defer it until later. We mention, however, that, since mice are fine structural etities, we shall iterate by Σ^*–extensions rather than the usual Σ_0–extensions. In the above examples, the minimal choice we made in our construction guaranteed that the mice we constructed were sound. However, in general we want the iterates of mice to themselves be mice. Thus we cannot require all mice to be sound: Suppose e.g. that $M = \langle J^E_\nu, F\rangle$ is a mouse and we form: $\pi : M \to^*_F M'$. Then M' is no longer sound. (To see this, let $p \in P^1_M$. It follows easily that $\pi(p) \in P^1_{M'}$. But $\kappa \notin \text{rng}(\pi)$; hence κ is not $\Sigma_1(M')$ in $\pi(p)$.)

As we said, however, our initial construction is designed to produce sound structures. Hence we *can* require that if $M = \langle J^E_\nu, F\rangle$ is a mouse and $\lambda < \nu$, then $M||\lambda$ is sound, since this property will not be changed by iteration.

By a *premouse* we mean a structure which has the salient properties of a mouse, but is not necessarily iterable. Putting our above remarks together, we arrive at the following definition:

Definition 3.3.4. $M = \langle J^E_\nu, F\rangle$ is a *premouse* iff it is acceptable and:

(a) Either $F = \emptyset$ or F is a full extender at a $\kappa < \nu$ with base $J_\tau[E]$, where $\tau = \kappa^{+M}$, and extension $\langle J_\nu[E], \pi\rangle$. Moreover F is weakly amenable and satisfies the initial segment condition. (Recall that $J = \langle J_\nu[E], E \cap J_\nu[E]\rangle$).

(b) Set $E_\gamma = E''\{\gamma\}$ for $\gamma < \nu$. If $\gamma < \nu$ is a limit ordinal, then $M||\gamma =: \langle J^E_\gamma, E_\gamma\rangle$ is sound and satisfies (a).

(c) $E = \{\langle x, \eta\rangle | x \in E_\eta \cap \eta < \nu \text{ is a limit ordinal}\}$.

We call a premouse $M = \langle J^E_\nu, F\rangle$ *active* iff $F \notin \emptyset$. If F is inactive we often write J^E_ν for $\langle J^E_\nu, \emptyset\rangle$. We classify active premice into three *types*:

Definition 3.3.5. Let F be an extender on κ with base S and extension $\langle S', \pi \rangle$. We set:

- $C = C_F =: \{\lambda | \kappa < \lambda < \pi(\kappa) \wedge F|\lambda \text{ is full}\}$
- F is of *type* 1 iff $C = \emptyset$
- F is of *type* 2 iff $C \neq \emptyset$ but is bounded in $\pi(\kappa)$
- F is of *type* 3 iff C is unbounded in $\pi(\kappa)$
- Let $M = \langle J^E_\nu, F \rangle$ be a premouse. The *type of* M is the type of F. We also set: $C_M =: C_F$.

It is evident that F satisfies the initial segment condition iff $F|\lambda \in S'$ whenever $\lambda \in C_F$.

Premice of differing type will very often require different treatment in our proofs. In much of this book we will assume that there is no inner model with a Woodin cardinal, which implies that all mice are of type 1. For now, however, we continue to work in greater generality.

Lemma 3.3.1. *Let F be an extender at κ with base S and extension $\langle S', \pi \rangle$. Let $\kappa < \lambda < \pi(\kappa)$. Then $\lambda \in C_F$ iff $\pi(f)(\alpha_1, \ldots, \alpha_n) < \lambda$ for all $f \in M$ such that $f : \kappa^n \to \kappa$ and all $\alpha_1, \ldots, \alpha_n < \lambda$.*

Proof: We first prove the direction $(\rightarrow)$. Let $F^* = F|\lambda$ be full with extension $\langle S^*, \pi^* \rangle$. Let $f, \alpha_1, \ldots, \alpha_n$ be as above. Let $\beta = \pi^*(f)(\vec{\alpha})$. Set $e = \{\langle \xi_1, \ldots, \xi_n, \delta \rangle | f(\vec{\xi}) = \delta\}$. Then $\beta < \lambda$ and:

$$\langle \vec{\alpha}, \beta \rangle \in F^*(e) = \lambda^{n+1} \cap F(e).$$

Hence $\pi(f)(\vec{\alpha}) = \beta < \lambda$. QED $(\rightarrow)$

We now prove $(\leftarrow)$. Let $f, \alpha_1, \ldots, \alpha_n$ be as above. Then $\pi(f)(\vec{\alpha}) = \beta < \lambda$. Hence

$$\langle \vec{\alpha}, \beta \rangle \in F(e) \cap \lambda^{n+1} = F^*(e).$$

Hence $\pi^*(f)(\vec{\alpha}) = \beta < \lambda$. But each $\gamma < \pi^*(\kappa)$ has the form $\pi^*(f)(\vec{\alpha})$ for some such $f, \alpha_1, \ldots, \alpha_n < \lambda$. Hence $\pi^*(\kappa) = \lambda = \text{length } (F^*)$.

QED (Lemma 3.3.1)

Corollary 3.3.2. *C_F is closed in $\pi(\kappa)$.*

Corollary 3.3.3. *Let F, S, S', π be as above and let F be weakly amenable. Then C_F is uniformly $\Pi_1(\langle S', F \rangle)$ in κ.*

Proof: S' is admissible and the Gödel function $\prec, \succ$ is uniformly Σ_1 over admissible structures. By weak amenability we know that $\mathbb{P}(\kappa^2) \cap S = \mathbb{P}(\kappa^2) \cap S'$. S' is admissible and Gödel's pair function $\prec, \succ$ is $\Sigma_1(S')$ and defined on $(\mathrm{On}_{S'})^2$. Then "λ is Gödel–closed" is $\Delta_1(S')$, since it is expressed by $\bigwedge \xi, \delta < \lambda \prec \xi, \delta \succ < \lambda$. By Lemma 3.3.1, "$\lambda \in C_F$" is equivalent in S' to:

$$\kappa < \lambda \subset \pi(\kappa) \wedge \lambda \text{ is Gödel–closed}$$
$$\wedge \bigwedge f : n \to \kappa \bigwedge \alpha < \lambda \bigvee \beta < \lambda \prec \alpha, \beta \succ \in F(e_f)$$

where $e_f = \{\prec \delta, \xi \succ < \kappa | f(\xi) = \delta\}$. The function $f \mapsto e_f$ is $\Sigma_1(S')$ in κ and defined on $\{f \in S | f : \kappa \to \kappa\}$. Note that $\mu = \pi(\kappa)$ is expressible over $\langle S', F\rangle$ by $\langle \mu, \kappa\rangle \in F$ and $e' = F(e)$ is expressible by $\langle e', e\rangle \in F$. Thus $\lambda \in C_F$ is equivalent to the conjunction of 'λ is Gödel–closed' and:

$$\bigwedge e, e', \mu, f(((\langle e', e\rangle \in F \wedge \langle \mu, \kappa\rangle \in F \wedge f : \kappa \to \kappa \wedge e = e_f)$$
$$\to (\kappa < \lambda < \mu \wedge \bigwedge \alpha < \lambda \bigvee \beta < \lambda \prec \alpha, \beta \succ \in e'))$$

QED (Lemma 3.3.3)

We now turn to the task of analyzing the complexity of the property of being a premouse and the circumstances under which this property is preserved by an embedding $\sigma : M \to M'$. If $M = \langle J^E_\nu, F\rangle$ is an active premouse, the answer to these question can vary with the type of F.

We shall be particularly interested in the case that, for some weakly amenable extender G on M at a $\tilde{\kappa} < \rho^n_M$, M' is the $\Sigma^{(n)}_0$ extension $\langle M', \sigma\rangle$ of M by G (i.e. $\sigma : M \to^{(n)}_G M'$). In this case we shall prove:

- M' is a premouse
- If M is active, then M' is active and of the same type
- If M is of type 2, then $\sigma(\max C_M) = \max C_{M'}$.

This will be the content of Theorem 3.3.22 below. Note that if G is close to M in the sense of §3.2, and n is maximal with $\tilde{\kappa} < \rho^n_M$, then M' is a fully Σ^*–preserving ultrapower of M (i.e. $\sigma : M \to^*_G M'$). In later sections we shall consider mainly iterations of premice by Σ^*–ultrapowers.

Note. In later sections we shall mainly restrict ourselves to premice of type 1. For the sake of completeness, however, we here prove the above result in full generality. The proof will be arduous.

We first define:

Definition 3.3.6. $M = \langle J_\nu^E, F\rangle$ is a *mouse precursor (or precursor for short) at* κ iff the following hold:

- M is acceptable
- $\kappa \in M$ and $\tau = \kappa^{+M} \in M$
- F is a full extender at κ on J_τ^E with extension $\langle J_\nu^E, \pi\rangle$.

Note. F then has base $J_\tau[E]$ and extension $\langle J_\nu[E], \pi\rangle$.

Note. F is weakly amenable, since $\mathbb{P}(\kappa) \cap M \subset J_\tau[E]$ by acceptability.

Lemma 3.3.4. *$M = \langle J_\nu^E, F\rangle$ is a precursor at κ iff the following hold:*

(a) M is acceptable

(b) F is a function defined on $\mathbb{P}(\kappa) \cap M$

(c) $F \upharpoonright \kappa = \mathrm{id}$, $\kappa < F(\kappa) = \lambda$, where λ is the largest cardinal in M.

(d) Let $a_1, \ldots, a_n \in \mathbb{P}(\kappa) \cap M$. Let φ be a Σ_1 forumla. Then:

$$J_\nu^E \models \varphi[\vec{a}] \leftrightarrow J_\nu^E \models \varphi[F(\vec{a})]$$

(e) Let $\xi < \nu$. There is $X \in \mathbb{P}(\kappa) \cap M$ such that

$$F(X) \notin J_\xi^E.$$

Proof: We first note that $J_\nu^E \models \varphi[\vec{a}]$ can be replaced by $J_\tau^E \models \varphi[\vec{a}]$ where $\tau = \kappa^{+M}$, by acceptability. The direction $(\rightarrow)$ then follows easily. We prove $(\leftarrow)$.

We first note that F injects $\mathbb{P}(\kappa) \cap M$ into $\mathbb{P}(\lambda) \cap M$. F is injective by (d). But if $X \subset \kappa$, then $F(X) \subset F(\kappa) = \lambda$ by (d).

(1) $J_\kappa^E \prec J_\lambda^E$.

Proof: We first recall that by §2.4 each $x \in J_\kappa^E$ has the form $f(a)$ for some first $a \subset \kappa$, where f is $\Sigma_1(J_\kappa^E)$. By §2.4 we can choose the Σ_1 definition of f as being functionally absolute in J–models. Now let $x_1, \ldots, x_n \in J_\kappa^E$.

Let φ be a first order formula. We claim:

$$J_\kappa^E \models \varphi[\vec{x}] \rightarrow J_\lambda^E \models \varphi[\vec{x}].$$

Let $x_i = f_i(a_i)$, where $a_i \subset \kappa$ is finite and f_i has a functionally absolute definition '$x = f_i(a)$'. Then $J^E_\lambda \models$ '$x_i = f_i(a_i)$' for $i = 1, \ldots, n$. Let Ψ be the formula:

$$\bigvee x_1 \ldots x_n (\bigwedge_{i=1}^{n} x_i = f_i(a_i) \wedge \varphi(\vec{x})).$$

Then:

$$J^E_\kappa \models \varphi[\vec{x}] \leftrightarrow J^E_\kappa \models \Psi[\vec{a}]$$

and:

$$J^E_\lambda \models \varphi[\vec{x}] \leftrightarrow J^E_\lambda \models \Psi[\vec{a}].$$

But $J^E_\kappa \models \Psi[\vec{a}]$ is $\Sigma_1(M)$ in $\kappa, \vec{a}$ and $J^E_\lambda \models \Psi[\vec{a}]$ is $\Sigma_1(M)$ in $\lambda, \vec{a}$ by the same definition. Moreover $F(a_i) = a_i$ $(i = 1, \ldots, n)$ and $F(\kappa) = \lambda$.

Hence by (d):

$$\begin{aligned} J^E_\kappa \models \varphi[\vec{x}] &\leftrightarrow J^E_\kappa \models \Psi[\vec{a}] \\ &\leftrightarrow J^E_\lambda \models \Psi[\vec{a}] \\ &\leftrightarrow J^E_\lambda \models \varphi[\vec{x}]. \end{aligned}$$

QED (1)

It follows easily, using acceptability, that J^E_κ and J^E_λ are ZFC^- models. Gödel's pair function $\prec, \succ$ then has a uniform definition on J^E_κ and J^E_λ. Hence $\langle \prec \alpha, \beta \succ | \alpha, \beta \in J^E_\kappa \rangle$ is $\Sigma_1(M)$ in κ and $\langle \prec \alpha, \beta \succ | \alpha, \beta \in J^E_\lambda \rangle$ is $\Sigma_1(M)$ in λ by the same definition.

For any $X \subset \kappa$ there is at most one function $\Gamma = \Gamma_X$ defined on κ such that $\Gamma(\alpha) = \{\Gamma(\beta) | \langle \beta, \alpha \rangle \in X\}$ for $\alpha < \kappa$. For $X \in \mathbb{P}(\kappa) \cap M$ the statement $f = \Gamma_X$ is uniformly $\Sigma_1(M)$ in X, f, κ. Moreover the statement $\bigvee f\ f = \Gamma_X$ ('Γ_X is defined') is uniformly $\Sigma_1(M)$ in X, κ. The same is true at λ: For $Y \subset \lambda$ the statement $f = \Gamma_Y$ is uniformly $\Sigma_1(M)$ in Y, f, λ and the statement $\bigvee f\ f = \Gamma_Y$ is uniformly $\Sigma_1(M)$ in Y, λ by the same definition.

We must define a π such that $\langle J_\nu[E], \pi \rangle$ is the extension of F. The above remarks suggest a way of doing so:

Definition 3.3.7. Let $x \in J^E_\tau$, $x \in u$, where $u \in J^E_\tau$ is transitive. Let $f \in J^E_\tau$ map κ onto u. Set:

$$X =: \{\prec \alpha, \beta \succ | f(\alpha) \in f(\beta)\},$$

then $f = \Gamma_X$. Let $f' =: \Gamma_{F(X)}$. Let $x = f(\xi)$ where $\xi < \kappa$. Set:

$$\pi(x) = \pi_{f,\xi}(x) =: f'(\xi).$$

We must first show that π is independent of the choice of f, ξ. Suppose that $x \in v$, where $v \in J^E_\tau$ is transitive, and $g \in J^E_\tau$ maps κ onto v. Then, letting $Y = \{\prec \alpha, \beta \succ | g(\alpha) \in g(\beta)\}$, we have: Let $x = g(\zeta)$. Then by (d):

$$f(\xi) = \Gamma_X(\xi) = \Gamma_Y(\zeta) \to \pi_{f,\xi}(x) = \Gamma_{F(X)}(\xi) = \Gamma_{F(Y)}(\zeta) = \pi_{g,\zeta}(x).$$

Similarly we get:

(2) $\pi : J^E_\tau \to_{\Sigma_0} j^E_\nu$.

Proof: Let $x_1, \ldots, x_n \in J^E_\tau$. Let $x_1, \ldots, x_n \in u$, where $u \in J^E_\tau$ is transitive. Let $f_i \in J^E_\tau$ map κ onto $u (i = 1, \ldots, n)$. Set: $X_i = \{\prec \alpha, \beta \succ | f_i(\alpha) \in f_i(\beta)\}$. Let $x_i = f_i(\xi_i)$. Let φ be Σ_0. By (d) we conclude:

$$\begin{aligned} J^E_\tau \models \varphi[\vec{x}] \quad &\leftrightarrow J^E_\tau \models \varphi(\Gamma_{\vec{X}}(\vec{\xi})) \\ &\leftrightarrow J^E_\tau \models \varphi(\Gamma_{F(\vec{X})}(\vec{\xi})) \end{aligned}$$

where $F(X_i)(\xi_i) = \pi(\xi_i)$. QED (2)

(3) $F(X) = \pi(X)$ for $X \in \mathbb{P}(\kappa) \cap M$.

Proof: Let $X = f(\mu)$ where $\mu < \kappa$, $f \in J^E_\tau$, and $f : \kappa \to u$, where u is transitive. Set: $Y =: \{\prec \alpha, \beta \succ | f(\alpha) \in f(\beta)\}$. Then $f = \Gamma_Y$ and $X = \Gamma_Y(\mu)$. By (d) we conclude:

$$F(X) = \Gamma_{F(Y)}(\mu) = \pi(X).$$

QED (3)

It remains only to show:

(4) $\pi : J^E_\tau \to J^E_\nu$ cofinally.

Proof: Let $y \in J^E_\nu$. If $y \in J^E_\xi$, $\xi < \nu$, there is an $X \in \mathbb{P}(\kappa) \cap M$ such that $F(X) \notin J^E_\xi$. Let $X \in J^E_\mu$, $\mu < \tau$. Then:

$$F(X) = \pi(X) \in J^E_{\pi(\mu)}.$$

Hence $\pi(\mu) > \xi$ and:

$$y \in J^E_{\pi(\mu)} = \pi(J^E_\mu).$$

QED (Lemma 3.3.4)

Corollary 3.3.5. *Let* $M = \langle J^E_\nu, F\rangle$*. The statement 'M is a precursor' is uniformly* $\Pi_2(M)$*.*

Proof: The conjunction of (a) – (e) is uniformly $\Pi_2(M)$ in the parameters κ, λ. Let it have the form $R(\kappa, \lambda)$, where R is Π_2. It is evident that if $R(\kappa, \lambda)$ holds, then $\langle\kappa, \lambda\rangle$ is the unique pair of ordinals which is an element of F. Hence the conjunction (a) – (e) is expressible by:

$$\bigvee \kappa, \lambda\langle\kappa, \lambda\rangle \in F \wedge \bigwedge \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \to R(\kappa, \lambda)).$$

QED (Corollary 3.3.5)

Definition 3.3.8. $M = \langle J^E_\nu, F\rangle$ is a *good precursor* iff M is a precursor and F satisfies the initial segment condition.

Corollary 3.3.6. *Let* $M = \langle J^E_\nu, F\rangle$*. The statement 'M is a good precursor at* κ*' is uniformly* $\Pi_3(M)$*.*

Proof: Let M be a precursor. Then F satisfies the initial segment condition iff in M we have, letting $C =: C_F$:

$$\begin{aligned}&\bigwedge \eta \in C \bigvee F'(F' \text{ is a function } \wedge \operatorname{dom}(F) = \mathbb{P}(\kappa))\\ &\wedge \bigwedge Y, X(\langle Y, X\rangle \in F \to \langle Y \cap \eta, X\rangle \in F')\end{aligned}$$

This is Π_3 since C is Π_2. QED (Lemma 3.3.6)

Lemma 3.3.7. *Let* $M = \langle J_\nu, F\rangle$ *be a precursor at* κ*. Let* $\tau = \kappa^{+M}$ *and let* $\langle J^E_\nu, \pi\rangle$ *be the extension of* J^E_τ *by* F*. Then* π *and* $\operatorname{dom}(\pi)$ *are uniformly* $\Delta_1(M)$*.*

Proof: π is uniformly $\Sigma_1(M)$ in κ, λ since by the definition of π in the proof of Lemma 3.3.4 we have:

$$\begin{aligned}y &= \pi(x) \leftrightarrow \bigvee f \bigvee u \bigvee X \bigvee \xi \bigvee Y(u \text{ is transitive } \wedge\\ &f : \kappa \xrightarrow{\text{onto}} u \wedge x = f(\xi) \wedge X = \{\prec \alpha, \beta \succ | f(\alpha) \in f(\beta)\}\\ &\wedge Y = F(X) \wedge y = \Gamma_Y(\xi)).\end{aligned}$$

Let $\varphi(\kappa, \lambda, y, x)$ be the uniform Σ_1 definition of π from κ, λ. Then $\langle\kappa, \lambda\rangle$ is the unique pair of ordinals such that $\langle\kappa, \lambda\rangle \in F$. Hence:

$$y = \pi(x) \leftrightarrow \bigvee \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \wedge M \models \varphi[\kappa, \lambda, y, x]).$$

Then π is uniformly $\Sigma_1(M)$. But $\operatorname{dom}(\pi) = J^E_\tau$; hence:

$$\begin{aligned}y \in \operatorname{dom}\pi \quad &\leftrightarrow \bigvee \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \wedge y \in (J^E_{\kappa^+})^{J^E_\lambda})\\ &\bigwedge \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \to y \in (J^E_{\kappa^+})^{J^E_\lambda}).\end{aligned}$$

Thus $\text{dom}(\pi)$ is uniformly $\Delta_1(M)$. But then

$$\begin{aligned} y = \quad & \pi(x) \leftrightarrow (y \in \text{dom}(\pi) \wedge \\ & \bigwedge y' \in M(y \neq y' \to y' \neq \pi(x))). \end{aligned}$$

Thus π is $\Delta_1(M)$. QED (Lemma 3.3.7)

But then:

Corollary 3.3.8. *Let $\sigma : M \to_{\Sigma_1} M'$ where $M = \langle J_\nu^E, F\rangle$ and $M' = \langle J_{\nu'}^{E'} F'\rangle$ are precursors. Let $\langle J_\nu^E, \pi\rangle$ be the extension of J_τ^E by F and $\langle J_{\nu'}^{E'}, \pi'\rangle$ be the extension of $J_{\tau'}^{E'}$ by F. Then:*

$$\sigma\pi(x) \simeq \pi'\sigma(x) \text{ for } x \in M.$$

The satisfaction relation for an amenable structure $\langle J_\nu^E, B\rangle$ is uniformly $\Delta_1(M)$ in the parameter $\langle J_\nu^E, B\rangle$ whenever $M \ni \langle J_\nu^E, B\rangle$ is transitive and rudimentarily closed.

(To see this note that, letting $E = E \cap J_\nu^E$, the structure $\langle M, E, B\rangle$ is rud closed. Hence its Σ_0–satisfaction is $\Delta_1(\langle M, E, B\rangle)$ or in other words $\Delta_1(M)$ in E, B. But if φ is any formula in the language of $\langle J_\nu^E, B\rangle$, we can convert it to a Σ_0 formula $\overline{\varphi}$ in the language of $\langle M, E, B\rangle$ simply by bounding all quantifiers by a new variable v. Then:

$$\langle J_\nu^E, B\rangle \models \varphi[\vec{x}] \leftrightarrow \langle M, E, B\rangle \models \overline{\varphi}[J_\nu[E], \vec{x}]$$

for all $x_1, \ldots, x_n \in J_\nu^E$.)

It is apparent from §2.5 that for each n there is a statement φ_n such that

$$\langle J_\nu^E, B\rangle \text{ is } n\text{–sound } \leftrightarrow \langle J_\nu^E, B\rangle \models \varphi_n.$$

Moreover the sequence $\langle \varphi_n | n < \omega\rangle$ is recursive. Thus

Lemma 3.3.9. *"$\langle J_\nu^E, B\rangle$ is sound" is uniformly $\Pi_1(M)$ in $\langle J_\nu^E, B\rangle$ for all transitive rud closed $M \ni \langle J_\nu, B\rangle$.*

Using this we get:

Lemma 3.3.10. *Let J_ν^E be acceptable. The statement '$\langle J_\nu^E, \emptyset\rangle$ is a premouse' is uniformly $\Pi_1(J_\nu^E)$.*

Proof: $\langle J_\nu^E, \emptyset\rangle$ is a premouse iff the following hold in J_ν^E:

- $\bigwedge x \in E \bigvee \nu, z \in TC(x)(x = \langle z, \nu\rangle \wedge \nu \in \text{Lm} \wedge z \in J_\nu^E)$

- $\bigwedge \nu(\nu \in \mathrm{Lm} \to \langle J_\nu^E, E''\{\nu\}\rangle$ is sound)
- $\bigwedge \nu(E''\{\nu\} \neq \emptyset \to \langle J_\nu^E, E''\{v\}\rangle$ is a good precursor).

QED (Lemma 3.3.10)

An immediate corollary is:

Corollary 3.3.11. *Let $\overline{M}, M$ be acceptable. Then:*

- *If $\pi : \overline{M} \to_{\Sigma_1} M$ and $\overline{M}$ is a passive premouse, then so is M.*
- *If $\pi : \overline{M} \to_{\Sigma_0} M$ and M is a passive premouse, then so is $\overline{M}$.*

The property of being an active premouse will be harder to preserve. $\langle J_\nu^E, F\rangle$ is an active premouse iff $\langle J_\nu^E, \emptyset$ is a passive premouse and $\langle J_\nu^E, F\rangle$ is a good precursor. Hence:

Lemma 3.3.12. *'$\langle J_\nu^E, F\rangle$ is an active premouse' is uniformly $\Pi_3(\langle J_\nu^E, F\rangle)$.*

Note. This uses that being acceptable is uniformly $\Pi_1(\langle J_\nu^E, F\rangle)$ when $\nu \in \mathrm{Lm}^*$.

An immediate, but not overly useful, corollary is:

Corollary 3.3.13. *Let $\overline{M}, M$, be J–models.*

- *If $\pi : \overline{M} \to_{\Sigma_3} M$ and $\overline{M}$ is an active premouse, then so is M.*
- *If $\pi : \overline{M} \to_{\Sigma_2} M$ and M is an active premouse, then so is $\overline{M}$.*

In order to get better preservation lemmas, we must think about the *type* of F in $\langle J_\nu^E, F\rangle$. F is of type 1 iff $C_F = \emptyset$. By Corollary 3.3.3 the condition $C_F = \emptyset$ is $\Pi_2(\langle J_\nu, F\rangle)$ uniformly. Hence

Lemma 3.3.14. *The statement 'M is an active premouse of type 1' is uniformly $\Pi_2(M)$ for $M = \langle J_\nu^E, F\rangle$.*

Hence

Corollary 3.3.15. *Let $\overline{M}, M$ be J–models.*

- *If $\pi : \overline{M} \to_{\Sigma_2} M$ and $\overline{M}$ is an active premouse of type 1, then so is M.*
- *If $\pi : \overline{M} \to_{\Sigma_1} M$ and M is an active premouse of type 1, then so is $\overline{M}$.*

A more important theorem is this:

Lemma 3.3.16. *Let M be an active premouse of type 1. Let $M = \langle J^E_\nu, F\rangle$ where $\kappa = \mathrm{crit}(F)$. Let G be a weakly amenable extender on M at $\tilde{\kappa}$, where $\tilde{\kappa} < \rho^n_M$. Let $\langle M', \sigma\rangle$ be the $\Sigma^{(n)}_0$ extension of M by G. Then M' is an active premouse of type 1.*

Proof: We consider two cases:

Case 1 $n = 0$.

Claim 1 $M' = \langle J^{E'}_{\nu'}, F'\rangle$ is a precursor.

(1) F' is a function and $\mathrm{dom}(F') \subset \mathbb{P}(\kappa)$, since these statements are Π_1 and σ is Σ_1 preserving
For $\xi < \tau = \kappa^{+M}$ set: $\pi[\xi] = \pi \restriction J^E_\xi, \pi'[\xi] = \sigma(\pi[\xi])$, then

(2) $\pi'[\xi] : J^E_{\sigma(\xi)} \prec J^E_{\sigma\pi(\xi)}$,
since $\pi[\xi] : J^E_\xi \prec J^E_{\pi(\xi)}$.
Set: $\pi' = \bigcup\limits_\xi \pi'[\xi]$. Since $\sup \pi''\tau = \nu$ and $\sup \sigma''\nu = \nu'$, we have

(3) $\sigma : \langle M, \pi\rangle \to_{\Sigma_0} \langle M', \pi'\rangle$ cofinally.

(4) $\mathrm{dom}(\pi') = \bigcup\limits_{\xi<\tau} \tau(J^E_\xi) = J^{E'}_{\tau'}$,
where $\tau' = \sigma(\tau) = \kappa'^{+M'}$ and $\kappa' = \sigma(\kappa)$. Hence

(5) $\pi' : J^{E'}_{\tau'} \to_{\Sigma_0} J^{E'}_{\nu'}$ cofinally.

(6) $F' = \pi' \restriction \mathbb{P}(\kappa')$
by (3) and:

$$\bigwedge X(X \in J^E_{\sigma(\xi)} \cap \mathbb{P}(\kappa') \to \langle \pi'(X), X\rangle \in F'),$$

since the corresponding Π_1 statement holds of ξ in M.

It follows easily that $\langle J_{\nu'}[E'], \pi'\rangle$ is the extension of $J^E_{\tau'}$ by F'.

QED (Claim 1)

Claim 2 F' is of type 1 (hence F' satisfies the initial segment condition).

Proof: Let $\xi < \lambda' = \pi'(\kappa')$. Using Lemma 3.3.1 we show:

Claim $\xi \notin C_{F'}$.

Let $\zeta \in M$ be least such that $\sigma(\zeta) \geq \zeta$. Since $\zeta \notin C_F$, there is $f : \kappa^n \to \kappa$ in M such that $\pi(f)(\vec{\alpha}) > \zeta$ for some $\alpha_1, \ldots, \alpha_n < \zeta$. But then $\sigma(\alpha_1), \ldots, \sigma(\alpha_n) < \xi$ and

$$\pi'(\sigma(f))(\sigma(\vec{\alpha})) = \sigma(\pi(f))(\vec{\alpha})) > \sigma(\zeta) \geq \xi.$$

Hence $\xi \notin C_{F'}$. QED (Claim 2)

Thus $J^{E'}_{\nu'}$ is a premouse by Corollary 3.3.11 and M' is a good precursor of type 1. Hence M' is a premouse of type 1. QED (Case 1)

Case 2 $n > 1$.

Then σ is Σ_2–preserving by Lemma 3.2.12. Hence M' is a premouse of type 1 by Corollary 3.3.15 QED (Corollary 3.3.16)

We now consider premice of type 2. $M = \langle J^E_\nu, F\rangle$ is a premouse of type 2 iff J^E_ν is a premouse, M is a precursor and $F|\eta \in J^E_\nu$ where $\eta = \max C_F$. (It then follows that $F|\mu = (F|\eta)|\mu \in J^E_\nu$ whenever $\mu \in C_F$.) The statement $e = F|\mu$ is uniformly $\Pi_1(M)$ in e, u, μ, since it says:

$$e \text{ is a function } \wedge \bigwedge x \in \mathbb{P}(\kappa) \cap M e(X) = F(X) \cap \mu.$$

But then the statement:

$$e = F|\eta \wedge \eta = \max C_F$$

is $\Pi_2(M)$ in e, η, κ uniformly, since it says: $e = F|\eta \wedge C_F \setminus \eta = \emptyset$, where C_F is uniformly $\Pi_2(M)$. It then follows easily that:

Lemma 3.3.17. *Let $M = \langle J^E_\nu, F\rangle$, $M = \langle J^{E'}_{\nu'}, \overline{F}\rangle$.*

- If $\pi : \overline{M} \to_{\Sigma_2} M$ and $\overline{M}$ is a premouse of type 2, then so is M. Moreover, $\pi(\max C_E) = \max C_F$.
- If $\pi : \overline{M} \to_{\Sigma_1} M$, M is a premouse of type 2 and $e = F|\max(C_F) \in \operatorname{rng}(\pi)$, then $\overline{M}$ is a premouse of type 2 and $\pi(\max C_{\overline{F}}) = \max C_F$.

We also get:

Lemma 3.3.18. *Let M be a premouse of type 2. Let G be a weakly amenable extender on M at $\tilde{\kappa}$, where $\tilde{\kappa} < \rho^n_M$. Let $\langle M', \sigma\rangle$ be the $\Sigma^{(n)}_0$ extension of M by G. Then M' is a premouse of type 2. Moreover, $\sigma(\max C_M) = \max C_{M'}$.*

Proof: If $n > 0$, then σ is Σ_2–preserving and the result follows by Lemma 3.3.17. Now let $n = 0$. Let $M = \langle J^E_\nu, F\rangle$ where F is an extender at κ on J^E_τ (where $\tau = \kappa^{+M}$. Let $M' = \langle J^{E'}_{\nu'}, F'\rangle$). It follows exactly as in Lemma 3.3.16 that $J^{E'}_{\nu'}$ is a premouse and M' is a precursor. We must prove:

Claim F' is of type 2. Moreover, $\tau(\max C_F) = \max C_{F'}$.

Proof: Let $\eta = \max C_F$, $e = F|\eta$. Then $\sigma(e) = F'|\eta'$, since this is a Π_1 condition. But then $C_{F'} \setminus \eta' = \emptyset$ follows exactly as in Lemma 3.3.16, since $C_F \setminus \eta = \emptyset$ and σ takes $\lambda = F(\kappa)$ cofinally to $\lambda' = F'(\kappa')$.

QED (Lemma 3.3.18)

We now turn to premice of type 3. One very important property of these structures is:

Lemma 3.3.19. *Let $M = \langle J^E_\nu, F\rangle$ be a premouse of type 3. Let $\lambda = F(\kappa)$ where F is at κ. Then $\rho^1_M = \lambda$.*

Proof:

(1) $h_M(\lambda) = M$ (hence $\rho^1_M \leq \lambda$).

Proof: Note that if $X \in \mathbb{P}(\kappa) \cap M$, then $X \in J^E_\tau \subset h_M(\tau)$. Hence $F(X) \in h_M(\tau)$. Now let $\langle J^E_\nu, \pi\rangle$ be the extension of J^E_τ by F. Then $\pi''\tau$ is cofinal in ν. But $\pi''\tau \subset h_M(\tau)$, since if $f \in M, f : \kappa \leftrightarrow \eta$, and $X = \{\prec \xi, \zeta \succ | f(\xi) < f(\zeta)\}$, then $F(X) = \{\prec \xi, \zeta \succ |\pi(f)(\xi) < \pi(f)(\zeta)\}$, where $\pi(f) : \lambda \leftrightarrow \pi(\eta)$. Hence $\pi(\eta) = \text{otp}(F(X)) \in h_M(\tau)$. But iff $g =$ the J^E_ν–least $g : \lambda \overset{\text{onto}}{\longrightarrow} \pi(\eta)$, then $g \in h_M(\tau)$. Hence $\pi(\eta) = g''\lambda \subset h_M(\lambda)$ for all $\eta < \tau$. Hence $\nu \subset h_M(\lambda)$. QED (1)

(2) Let $D \subset \lambda$ be $\underline{\Sigma}_1(M)$. Then $\langle J^E_\lambda, D\rangle$ is amenable. (Hence $\rho^1_M \geq \lambda$.)

Proof: By (1) D is $\Sigma_1(M)$ in a parameter $\alpha < \lambda$. Let $\eta \in C_F$ such that $\eta > \alpha$. Then $E = F|\eta \in M$. Since J^E_λ is a ZFC^- model, we have:

$$\langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle \in J^E_\lambda, \text{ where } \pi : J^E_\tau \to_{\overline{F}} J^{\overline{E}}_{\overline{\nu}}.$$

We then observe that there is a unique $\sigma : J^{\overline{E}}_{\overline{\nu}} \prec J^E_\nu$ defined by

$$\sigma(\overline{\pi}(f)(\beta)) = \pi(f)(\beta) \text{ for}$$
$$f \in J^E_\tau, f : \kappa \to J^E_\tau, \beta < \eta.$$

Moreover, $\sigma \restriction \eta = \text{id}$ and σ is cofinal.

(To see that this definition works, let $\beta_1, \ldots, \beta_n < \eta,\ f_1, \ldots, f_n \in \tau$ such that $f_i : \kappa \to J^E_\tau$ for $i = 1, \ldots, n$. Set:

$$X = \{\prec \xi_1, \ldots, \xi_n \succ | J^E_\tau \models \varphi[f_1(\xi_1), \ldots, f_n(\xi_n)]\}.$$

Then:

$$\begin{aligned} J^{\overline{E}}_{\overline{\nu}} \models \varphi[\overline{\pi}(\vec{f}(\vec{\beta})] &\leftrightarrow \prec \vec{\beta} \succ \in \overline{F}(X) = \eta \cap F(X) \\ &\leftrightarrow J^E_\nu \models \varphi[\pi(\vec{f})(\vec{\beta})].) \end{aligned}$$

But $\sigma(\langle \overline{F}(Z), Z\rangle) = \langle F(Z), Z\rangle$ for $Z \in \mathbb{P}(\kappa) \cap M$. Hence:

$$\sigma(\overline{F} \cap U) = \sigma''(\overline{F} \cap U) = F \cap U.$$

By this we get:

$$\sigma : \langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle \to_{\Sigma_0} \langle J^E_\nu, F\rangle \text{ cofinally.}$$

Thus $\overline{D} = D \cap \eta$ is $\Sigma_1(\langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle)$ in α by the same definition as D over $\langle J^E_\nu, F\rangle$. Hence $\overline{D} \in J^E_\lambda$, since $\langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle \in J^E_\nu$. QED (Lemma 3.3.19)

If $M = \langle J^E_\nu, F\rangle$ is a precursor, then "F is of type 3" is uniformly $\Pi_3(M)$ in κ, since it is the conjunction:

$$\bigwedge \xi < \lambda \bigvee \eta < \lambda \cdot \eta \in C_F \wedge \bigwedge \xi < \eta \in C_F \bigvee e \in J^E_\lambda e = F|\eta.$$

Hence:

Lemma 3.3.20. *(a) Let $\pi : \overline{M} \longrightarrow_{\Sigma_3} M$ where $\overline{M}$ is a premouse of type 3. Then so is M.*

(b) Let $\pi : \overline{M} \longrightarrow_{\Sigma_2} M$ where M is a premouse of type 3. Then so is $\overline{M}$.

We also get:

Lemma 3.3.21. *Let $M = \langle J^E_\nu, F\rangle$ be a premouse of type 3. Let G be a weakly amenable extender at $\tilde{\kappa}$ on M. Let $\tilde{\kappa} < \rho^n_M$ and let $\langle M', \sigma\rangle$ be the $\Sigma^{(n)}_0$ extension of M by G. Then M' is a premouse of type 3.*

Proof: Let $M' = \langle J^{E'}_{\nu'}, F'\rangle$. We consider three cases:

Case 1 $n = 0$.

Exactly as in the previous lemmas we get: $J^{E'}_{\nu'}$ is a premouse and M' is a precursor. We must show:

Claim F is of type 3.

We know that σ takes λ cofinally to λ'. Let $\eta < \lambda, \eta \in C_F$. Let $e = F|\eta \in M$. Then $\sigma(\eta) \in C_{F'}$ and $\sigma(e) = F'|\sigma(\eta)$, since these statements are Π_1. Hence if $\mu < \lambda'$ there is $\eta \in C_F$ such that $\mu \leq \sigma(\eta)$ and

$$F'|\mu = (F'|\sigma(\eta))|\mu \in J^{E'}_{\lambda'}.$$

QED (Case 1)

Case 2 $n = 1$.

Then σ is Σ_2–preserving. Hence $J^{E'}_{\nu'}$ is a premouse and M' is a precursor. Let $\langle M, \pi\rangle$ be the extension of J^E_τ by F and $\langle M', \pi'\rangle$ the extension of $J^{E'}_{\tau'}$ by F', where $\tau = \kappa^{+M}, \tau' = \sigma(\tau) = \kappa'^{+M'}$.

We know that:

$$\sigma \upharpoonright J^E_\lambda : J^E_\lambda \to_G J^E_{\rho'},$$

where $\lambda = \pi(\kappa) = \rho^1_M$ and $\rho' = \sup \sigma"\lambda = \rho^1_{M'}$. Since τ is a successor cardinal in J^E_λ, we have $\tau \neq \mathrm{crit}(G)$. But then $\tau' = \sup \sigma"\tau$ by Lemma 3.2.6 of §3.2. π takes τ cofinally to ν and π' takes τ' cofinally to ν'. Using this we see:

(1) $\nu' = \sup \sigma"\nu$.

Proof: Let $\xi < \nu'$. Let $\zeta < \tau'$ such that $\pi'(\zeta) > \xi$. Let $\eta < \tau$ such that $\sigma(\eta) > \zeta$. By Corollary 3.3.8 we have:

$$\sigma\pi(\eta) = \pi'\sigma(\eta) > \xi.$$

QED (1)

But then it suffices to show:

Claim $\sigma : M \to_G M'$,

since then we can argue as in Case 1.

Let $x \in M'$. Let $\tilde{\kappa} = \text{crit}(\pi)$. We must show that $x = \sigma(f)(\xi)$ for an $f \in M$ such that $f : \kappa \to M$. Since M' is the $\Sigma_0^{(1)}$–ultrapower, we know:

$$x = \sigma(f)(\xi), \text{ where } f : \kappa \to M \text{ is } \underset{\sim}{\Sigma}_1(M).$$

Choosing a functionally absolute definition for f we have:

$$v = f(w) \leftrightarrow \bigvee y A(y, v, w, p)$$

where A is $\Sigma_0(M)$ and $p \in M$. By functional absoluteness we have:

$$v = \sigma(f)(w) \leftrightarrow \bigvee y A'(\eta, v, w, \sigma(p))$$

where A' is $\Sigma_0(M')$ by the same definition. Let $A'(y, x, \xi, \sigma(p))$. Since σ takes M cofinally to M' there is $a \in M$ such that $y, x \in \sigma(a)$ and $\tilde{\kappa} \subset a$. Set:

$$g(\mu) = \begin{cases} x \text{ if } x \in a \wedge \bigvee y \in a A(y, x, \mu, p) \\ 0 \text{ if no such } x \text{ exists.} \end{cases}$$

Then $g \in M$, $g : \tilde{\kappa} \to M$ and $x = \sigma(g)(\xi)$. QED (Case 2)

Case 3 $n > 1$.

Then $\rho^1_{M'} = \tau(\rho^1_M) = \lambda'$ and σ is $\Sigma_2^{(1)}$–preserving by Lemma 3.2.12. But C_F is now $\Sigma_0^{(1)}(M)$ and $e = F|\eta$ is $\Sigma_0^{(1)}(M)$ for $e, \eta \in J^E_\lambda$. The statements:

$$\bigwedge \xi < \lambda \bigvee \eta < \lambda(\xi < \eta \in C_F, \ \bigwedge \eta \in C_F(\bigvee e \in J^E_\lambda \ e = F|\eta)$$

are now $\Pi_2^{(1)}(M)$. Hence the corresponding statements hold in M'. Hence $C_{F'}$ is unbounded in λ' and $F'|\eta \in J^{E'}_{\lambda'}$ for $\eta \in C_{F'}$. Then M' is of type 3. QED (Lemma 3.3.21)

Combining lemmas 3.3.11, 3.3.13, 3.3.18 and 3.3.21 we have:

Theorem 3.3.22. *Let M be a premouse. Let G be an extender at $\tilde{\kappa}$ on M where $\rho_M^n > \tilde{\kappa}$. Let $\langle M', \sigma\rangle$ be the $\Sigma_0^{(n)}$ extension of M by G. Then:*

- *M' is a premouse*
- *If M is active then M' is active and of the same type*
- *If M is of type 2, then*

$$\sigma(\max C_M) = \max C_{M'}.$$

In order to show that premousehood is preserved under iteration we shall also need:

Theorem 3.3.23. *Let M_0 be a premouse. Let $\pi_{ij} : M_i \to_{\Sigma_1} M_j$ for $i \leq j \leq \eta$, where:*

- *$\pi_{i,i+1} : M_i \to_{G_i}^{(n_i)} M_{i+1}$, where G_i is an extender at $\tilde{\kappa}_i$ on $G_i (i < \eta)$*
- *M_i is transitive and the π_{ij} commate*
- *If $\lambda \leq \eta$ is a limit ordinal, then $M_\lambda, \langle \pi_i | i < \lambda\rangle$ is the transitivized direct limit of $\langle M_i | i < \lambda\rangle, \langle \pi_{ij} | i \leq j < \lambda\rangle$.*

Then:

- *M_η is a premouse*
- *If M_0 is active, then M_η is active and of the same type as M_0*
- *If M_0 is of type 2, then $\pi_{0\eta}(C_{M_0}) = C_{M'_\eta}$.*

Proof: We proceed by induction on η. Thus the assertion holds at every $i < \eta$. The case $\eta = 0$ is trivial, as is $\eta = \mu + 1$ by Theorem 3.3.22. Hence we assume that η is a limit ordinal. We make the following observation:

(1) Let φ be a Π_3 formula. Let $i < \eta, x_1, \ldots, x_n \in M_i$ such that $M_j \models \varphi[\pi_{ij}(\vec{x})]$ for $i \leq j < \eta$. Then $M_\eta \models \varphi[\pi_{i\eta}(\vec{x})]$.

Proof: Let $y \in M_\eta$. Pick j such that $i \leq j < \eta$ and $y = \pi_{i\eta}(\overline{y})$. Then $M_j \models \Psi[\overline{y}, \pi_{ij}(\vec{x})]$, where $\varphi = \bigwedge v\Psi$. Hence $M_j \models \chi[\overline{z}, \overline{x}, \pi_{ij}(\vec{x})]$ for some $\overline{z}$, where $\Psi = \bigvee u\chi$. Hence $M_\eta \models \chi[z, y, \pi_{i\eta}(\vec{x})]$ where $z = \pi_{i\eta}(\overline{z})$, since $\pi_{j\eta}$ is Σ_1–preserving. QED (1)

Each M_i is a premouse for $i < \eta$. But this condition is uniformly $\Pi_3(M_i)$ by Lemma 3.3.12. Hence M_η is a premouse. If M_0 is of type 1, then $C_{M_i} = \emptyset$ for $i < \eta$. But this condition is uniformly $\Pi_2(M_i)$; Hence M_η is of type 1.

Now let M_0 be of type 2 and let $\mu_0 = \max C_{M_0}$. Then M_i is of type 2 and $\mu_i = \max C_{M_i}$ for $i < \eta$, where $\mu_i = \Pi_{0i}(\mu_0)$. Let $e_0 = F_0|\mu_0$ where $M_0 = \langle J^{E_0}_{\nu_0}, F_0\rangle$. Then $e_i = F_i|\mu_i$ for $i < \eta$, since $e = F|\mu$ is a Π_1 condition. Thus for $i < \rho$ each M_i satisfies the Π_2 condition in e_i, μ_i:

$$e_0 = F_i|\mu_i \wedge C_{F_i} \setminus \mu_i = \emptyset.$$

Hence M_η satisfies the corresponding condition. Hence M_η is of type 2 and $\mu_\eta = \max(C_\eta)$. Clearly $C_{M_i} = C_{F_i} \cup \{\max C_{M_i}\}$ for $i \leq \eta$. Hence $\pi_{ij}(C_{M_i}) = C_{M_i}$.

Now assume that M_0 is of type 3. Then each $M_i (i < \eta)$ satisfies the Π_3 condition:

$$\bigwedge \xi < \lambda_i \bigvee \zeta < \lambda_i (\xi < \zeta \in C_{M_i}),$$
$$\bigwedge \zeta \in C_{M_i} \bigvee e \in J^{E_i}_{\lambda_i} e = F_i|\zeta.$$

But then M_η satisfies the corresponding conditions. Hence M_η is of type 3.

QED (Theorem 3.3.23)

3.4 Iterating premice

3.4.1 Introduction

We have stated that a mouse will be an iterable premouse, but left the meaning of the term *"iterable"* and *"iteration"* vague. Iteration turns out, indeed, to be a rather complex notion. Let us begin with the simplest example. most logicians are familiar with the iteration of a structure $\langle M, U\rangle$, where M is, say, a transitive ZFC^- model and $U \in M$ is a normal ultrafilter on $\mathbb{P}(U) \cap M$. Set: $M_0 = M, U_0 = U$. Applying U_0 to M_0 gives the ultraproduct $\langle M_1, U_1\rangle$ and the extension $\Pi_{0,1} : \langle M_0, U_0\rangle \to \langle M_1, U_1\rangle$ by U_0. We then repeat the process at $\langle M_1, U_1\rangle$ to get $\langle M_2, U_2\rangle$ etc. After $1 + \mu$ repetitions we get an *iteration of length* μ, consisting of a sequence $\langle\langle M_i, U_i\rangle | i < \mu\rangle$ of models and a commutative sequence $\langle \pi_{ij} | i \leq j < \mu\rangle$ of iteration maps $\pi_{ij} : M_i \to M_j$. These sequences are characterized by the conditions:

- $\pi_{i,i+1} : \langle M_i, U_i\rangle \to \langle M_{i+1}, U_i\rangle$ is the extension by U_i.
- The π_{ij} commute — i.e. $\pi_{ij} = \mathrm{id}$ and $\pi_{ij}\pi_{hi} = \pi_{hj}$ for $h \leq i \leq j < \mu$.