Each M_i is a premouse for $i < \eta$. But this condition is uniformly $\Pi_3(M_i)$ by Lemma 3.3.12. Hence M_η is a premouse. If M_0 is of type 1, then $C_{M_i} = \emptyset$ for $i < \eta$. But this condition is uniformly $\Pi_2(M_i)$; Hence M_η is of type 1.

Now let M_0 be of type 2 and let $\mu_0 = \max C_{M_0}$. Then M_i is of type 2 and $\mu_i = \max C_{M_i}$ for $i < \eta$, where $\mu_i = \Pi_{0i}(\mu_0)$. Let $e_0 = F_0|\mu_0$ where $M_0 = \langle J^{E_0}_{\nu_0}, F_0\rangle$. Then $e_i = F_i|\mu_i$ for $i < \eta$, since $e = F|\mu$ is a Π_1 condition. Thus for $i < \rho$ each M_i satisfies the Π_2 condition in e_i, μ_i:

$$e_0 = F_i|\mu_i \wedge C_{F_i} \setminus \mu_i = \emptyset.$$

Hence M_η satisfies the corresponding condition. Hence M_η is of type 2 and $\mu_\eta = \max(C_\eta)$. Clearly $C_{M_i} = C_{F_i} \cup \{\max C_{M_i}\}$ for $i \leq \eta$. Hence $\pi_{ij}(C_{M_i}) = C_{M_i}$.

Now assume that M_0 is of type 3. Then each $M_i (i < \eta)$ satisfies the Π_3 condition:

$$\bigwedge \xi < \lambda_i \bigvee \zeta < \lambda_i (\xi < \zeta \in C_{M_i}),$$
$$\bigwedge \zeta \in C_{M_i} \bigvee e \in J^{E_i}_{\lambda_i} e = F_i|\zeta.$$

But then M_η satisfies the corresponding conditions. Hence M_η is of type 3.

QED (Theorem 3.3.23)

3.4 Iterating premice

3.4.1 Introduction

We have stated that a mouse will be an iterable premouse, but left the meaning of the term *"iterable"* and *"iteration"* vague. Iteration turns out, indeed, to be a rather complex notion. Let us begin with the simplest example. most logicians are familiar with the iteration of a structure $\langle M, U\rangle$, where M is, say, a transitive ZFC^- model and $U \in M$ is a normal ultrafilter on $\mathbb{P}(U) \cap M$. Set: $M_0 = M, U_0 = U$. Applying U_0 to M_0 gives the ultraproduct $\langle M_1, U_1\rangle$ and the extension $\Pi_{0,1} : \langle M_0, U_0\rangle \to \langle M_1, U_1\rangle$ by U_0. We then repeat the process at $\langle M_1, U_1\rangle$ to get $\langle M_2, U_2\rangle$ etc. After $1 + \mu$ repetitions we get an *iteration of length* μ, consisting of a sequence $\langle\langle M_i, U_i\rangle | i < \mu\rangle$ of models and a commutative sequence $\langle \pi_{ij} | i \leq j < \mu\rangle$ of iteration maps $\pi_{ij} : M_i \to M_j$. These sequences are characterized by the conditions:

- $\pi_{i,i+1} : \langle M_i, U_i\rangle \to \langle M_{i+1}, U_i\rangle$ is the extension by U_i.
- The π_{ij} commute — i.e. $\pi_{ij} = \text{id}$ and $\pi_{ij}\pi_{hi} = \pi_{hj}$ for $h \leq i \leq j < \mu$.

- If $\lambda < \mu$ is a limit ordinal, then $M, \langle \pi_{i\lambda} | i < \lambda \rangle$ is the direct limit of:

$$\langle M_i | i < \lambda \rangle, \langle M_{ij} | i \leq j < \lambda \rangle.$$

Now suppose we are given a structure $\langle M, S \rangle$ where $S = \{\langle X, \kappa \rangle | X \in U_\kappa\}$ and for each $\kappa \in M$, eiter $U_\kappa = \emptyset$ or else κ is a measurable cardinal in M and $U_\kappa \in M$ is a normal ultrafilter on $\mathbb{P}(\kappa) \wedge M$. An *iteration* of $\langle M, S \rangle$ then consists of sequences $\langle \langle M_i, S_i \rangle | i < \mu \rangle$, $\langle M_{ij} | i \leq j < \mu \rangle$ and $\langle \kappa_i | i + 1 < \mu \rangle$.

The first condition above is then replaced by:

> $\pi_{i,i+1} : \langle M_i, S_i \rangle \to \langle M_{i+1}, S_{i+1} \rangle$ is the extension by the ultrafilter
> $U_i = \{X | \langle X, \kappa_i \rangle \in S_i\}$

The other conditions remain unchanged. $\kappa_i | i + 1 \leq \mu \rangle$ is called the sequence of *indices*. κ_i must always be so chosen that U_i is an ultrafilter.

Note. Since we are allowed considerable leeway in the choice of the index κ_i, the purist may question whether the word "iteration" is still appropriate. In fact, the mathematical meaning of this word has rapidly changed as the structures to which it is applied have grown more complex.

An iteration is called *normal* iff the indices are increasing — i.e. $\kappa_i < \kappa_j$ for $i < j < \mu$.

We now attempt to apply these ideas to premice. Let M be a premouse. An iteration of length μ will yield a sequence $\langle M_i | i < \mu \rangle$ of premice. In passing from M_i to M_{i+1} we apply any of the extenders E_ν^M such that $M_i || \nu = \langle J_\nu^E, E_\nu \rangle$ is active. $v = \nu_i$ is then the i–th index. (It would be ambiguous to regard $\kappa_i = \text{crit}(E_{\nu_i})$ as the index, since M_i might have many extenders with this critical point.) In a normal iteration we have that, whenever $i < j$, then:

$$J_{\nu_i}^{E^{M_i}} = J_{\nu_i}^{E^{M_j}} \text{ and } \nu_i \text{ is a cardinal in } M_j.$$

(In fact, $\nu_i = \lambda_i^{+M_j}$, where $\lambda_i = E_{\nu_i}(\kappa_i)$ is inaccessible in M_j.) This follows easily by induction on j. It was originally envisaged that E_{ν_0} would be applied directly to M_i to get M_{i+1}. It turns out, however, that such iterations are unsuitable for may purposes. (In particular, they are unsuited to use in *comparison iteration*, which we shall describe below.) The problem is that $\kappa_i = \text{crit}(E_{\nu_i})$ could be much smaller than λ_i, where $\lambda_i = E_{\nu_i}(\kappa_i)$ is the largest cardinal in the model $J_{\nu_i}^{E^{M_i}}$. In particular, we might have $\kappa_i < \lambda_h$ for an $h < i$. Since λ_h is an inaccessible cardinal in M_i, it follows by acceptability that:

$$\mathbb{P}(\kappa) \cap M_i = \mathbb{P}(\kappa) \cap J_{\lambda_h}^{E^{M_h}} \subset M_h.$$

Hence it should be possible to apply E_{ν_i} to M_h rather than M_i. It turns out that it is most effective to apply E_{ν_i} to the *smalles place possible*: we apply it to $M_{T(i+1)}$, where

$$\begin{aligned} T(i+1) =:\ & \text{the least } h \text{ such that either } h = i \\ & \text{or } h < i \text{ and } \kappa_i < \lambda_h. \end{aligned}$$

This should give us

$$\pi_{h,i+1} : M_h \to M_{i+1}.$$

Here, however, we must deal with a second problem, which can arise even when $T(i+1) = i$. We know that E_{ν_i} is an extender at κ_i on $J^E_{\nu_i}$. Then $\mathbb{P}(\kappa_i) \cap J^{E^{M_i}}_{\nu_i} = \mathbb{P}(\kappa_i) \cap J^{E^{M_i}}_{\tau_i} = \mathbb{P}(\kappa_i) \cap J^{E^{M_h}}_{\tau_i}$, where $\tau_i = \kappa_i^{+J^E_{\nu_i}}$. But M_h might contain subsets of κ_i which do not lie in $J^E_{\tau_i}$ (hence τ_i is not a cardinal in M_h, by acceptability). E_{ν_i} is then only a partial function on M_h and cannot be applied to M_h. The resolution of this difficulty is to apply E_{ν_i} to the *largest possible segment* of M_h. We set:

$$\begin{aligned} M_i^* =:\ & M_h||\eta_h, \text{ where } \eta_i \leq \mathrm{On}_{M_h} \text{ is maximal such that} \\ & \tau_h \text{ is a cardinal in } M_h||\eta. \end{aligned}$$

By acceptability, $\mathbb{P}(\kappa_i) \cap M_i^* = \mathbb{P}(\kappa_i) \cap J^E_{\tau_i}$ and $\rho^\omega_{M_i^*} \leq \kappa_i$ if $\eta_i < \mathrm{On}_{M_h}$.

We then say that M_h *drops* (or *truncates*) to M_i^*, if $M_h \neq M_i^*$. $i+1$ is then called a *drop point* (or *truncation point*). $\pi_{h,i+1} : M_i^* \to M_{i+1}$ is then a *partial* map of M_h to M_{i+1}

This means that iteration is no longer a linear process. Previously π_{ij} was defined whenever $i \leq j < \mu$, μ being the length of the iteration. Now it is defined only when i is less than or equal to j in a tree T on μ. (We write $i \leq_T j$ for $i = j \vee iT_j$.) 0 is the unique minimal point of T. $T(i+1)$ is the unique T–predecessor of $i+1$. The π_{ij} are partial maps and we again have:

$$\pi_{ij} \cdot \pi_{hi} = \pi_{hj} \text{ for } h \leq_T i \leq_T j.$$

We will always have: $iT_j \to i < j$, but the converse may not hold. If $\mu = \omega$, these conditions completely define $T \subset \omega^2$. But how do we then extend the iteration to an iteration of length $\omega + 1$? Previously we simply took a transitivized direct limit of $\langle M_i | i < \omega \rangle$, $\langle \pi_{ij} | i \leq j < \omega \rangle$. Now we must first find a *branch* b in T which is cofinal in ω (i.e. $\sup b = \omega$). We also require that b have at most finitely may drop points. Pick any $i \in b$ such that $b \setminus i$ has no drop point. Then $\pi_{hj} : M_h \to M_j$ is a total map on M_h for $i \leq_T h \underset{T\,i}{\leq} \in b$. Form the direct limit:

$$M_b, \langle \pi_{h_i} | i \leq h \in b \rangle$$

of:

$$\langle M_h | i \leq h \in b \rangle,\ \langle \pi_{hj} | i \leq_T h \leq j \in b \rangle.$$

If M_b is well founded, we call b a *well founded branch* and take M_b are being transitive. We can then continue the iteration by setting:

$$M_\omega =: M_b; hT_\omega \leftrightarrow: h \in b \text{ for } h < \omega.$$

$\pi_{j\omega}$ is then defined for $i \leq_T j <_T \omega$. If hTi, we set $\pi_{h\omega} =: \pi_{j\omega} \cdot \pi_{hi}$.

The same procedure is applied at all limit points λ. We then have:

- λ is a limit point of T
- $T''\{\lambda\}$ is cofinal in λ
- $T''\{\lambda\}$ contains at most finitely many truncation points.

By now we have almost given a virtual definition of what is meant by a "normal iteration of a premouse". The only point left vague is what we mean by "applying" the extender E_{ν_i} to M_i^*. We shall, in fact, take the $\Sigma_0^{(n)}$–ultrapower:

$$\pi : M_i^* \to_{E_{\nu_i}}^{(n)} M_{i+1},$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{M_i^*}$.

3.4.2 Normal iteration

We are now ready to write out the formal definition of "normal iteration". We shall employ the following notational devices:

Definition 3.4.1. Let T be a tree. We set:

- $i <_T j \leftrightarrow: \circ T_j$
- $i \leq_T j \leftrightarrow: i = j \vee iT_j$
- $[i, j]_T =: \{h | i \leq_T h \leq_T j\}$ (similarly for $[i, j]_T$, $[i, j]_T$, $[i, j]_T$)
- $T(i)$ =: The immediate T–predecessor of i (if it exists).

We can now define:

Definition 3.4.2. Let M be a premouse. By a *normal iteration* of M of *length* μ we mean:

$$\langle\langle M_i|i<\mu\rangle,\langle\nu_i|i+1<\mu\rangle,\langle\pi_{ij}|i\leq_T j\rangle,T\rangle$$

where.

(a) T is a tree on μ such that $iT_j \to j < j$

(b) M_i is a premouse for $i<\mu$

(c) $\nu_i<\nu_j$ if $i<j$. Moreover $M_i||\nu_i=\langle J^E_{\nu_i},E_{\nu_i}\rangle$ with $E_{\nu_i}\neq\emptyset$. (We set: κ_i =: crit$(E_{\nu_i}),\tau_i$ =: $\kappa_i^+ J^E_{\nu_i}$, λ_i =: $E_{\nu_i}(\kappa_i)$ = the largest cardinal in $J^E_{\nu_i}$.)

(d) Let h be least such that $h=i$ or $h<i$ and $\kappa_i<\lambda_h$. Then $h=T(i+1)$ and $J^{E^{M_h}}_{\tau_i+1}=J^{E^{M_i}}_{\tau_i+1}$.

(e) π_{ij} is a partial map of M_i to M_j. Moreover $\pi_{ij}\circ\pi_{hi}=\pi_{hj}$ for $h\leq_T i\leq_T j$.

(f) Let $h=T(i+1)$. Set: $M^*_i=M_h||\eta_i$, where η_i is maximal such that τ_i is a cardinal in $M_h||\eta_i$. Then $\pi_{h,i+1}:M^*_i\to^{(n)}_{E^{M_i}_{\nu_i}} M_{i+1}$, where $n\leq\omega$ is maximal such that $\kappa_i<\rho^n_{M^*_i}$. (We call $i+1$ a *drop point* or *truncation point* iff $M^*_i\neq M_h$)

(g) If $k\leq_j$ and $(i,j]_T$ has no drop point, then $\pi_{ij}:M_i\to M_j$ is a total function on M_i.

(h) Let λ be a limit ordinal. Then $T''\{\lambda\}$ is club in λ and contains at most finitely many drop points. Moreover, if $iT\lambda$ and $(i,\lambda)_T$ is free of drops, then:
$$M_\lambda,\langle\pi_{j\lambda}|i\leq_T j<_T\lambda\rangle$$
is the transitivized direct limit of:
$$\langle M_j|i\leq_T j<_T\lambda\rangle,\langle\pi_{hj}|i\leq_T h\leq_T j<_T\lambda\rangle.$$

This completes the definition.

Lemma 3.4.1. *Let* $I=\langle\langle M_i\rangle,\langle\nu_i\rangle,\langle\pi_{i,j}\rangle,T\rangle$ *be a normal iteration. Then*

(a) $J^{E^{M_i}}_{\nu_i}=J^{E^{M_{i+1}}}_{\nu_i}$

(b) *In* M_{i+1}, λ_i *is inaccessible and* $\nu_i=\lambda_i^+$.

Proof: τ_i is a cardinal in M_i^*. Since κ_i is inaccessible in $J_{\tau_i}^{E^{M_i}}$ and is the largest cardinal in $J_{\tau_i}^{E^{M_i}}$, it follows by acceptability that:

$$\tau_i = \kappa_i^+ \text{ and } \kappa_i \text{ is inaccessible in } M_i^*$$

$F = E^{M_i}{}_{\nu_i}$ is a full extender of length λ_i with base $H = |J_{\tau_i}^{E^{M_i}}|$ and extension $\langle \pi, H' \rangle$, where $H' = |J_{\nu_i}^{E^{M_i}}|$. By acceptability we have:

$$\mathbb{P}(\kappa_i) \cap M_i^* = \mathbb{P}(\kappa_i) \cap J_{\tau_i}^{E^{M_i}}$$

Hence F is an extender on M_i^* (and the condition (f) makes sense). But then $\langle M_{i+1}, \pi_{i,i+1} \rangle$ is the $\Sigma_i^{(n)}$-liftup of $\langle M_i^*, \pi \rangle$, where n is maximal such that $\kappa_i < \rho^n_{M_i^*}$. Hence:

$$\pi_{i,i+1}(\tau_i) = \sup \pi\text{"}\tau_i = \nu_i \text{ and } \pi_{i,i+1}(\kappa_i) = \lambda_i$$

Hence (b) holds, since the corresponding statement is function of κ_i, τ_i in M_i^*.

To see that (a) holds, note that each element of H' has the form $\pi(f)(\alpha)$, where $\alpha < \lambda_0$ and $f \in H$ is a function on κ. But then:

$$\pi(f)(\alpha) \in E^{M_i} \longleftrightarrow \pi(f)(\alpha) \in E^{M_{i+1}} \longleftrightarrow \alpha \in \pi(X)$$

where $X = \{\xi < \kappa_i : f(\xi) \in E^{M_i}\}$. Hence

$$E^{M_i} \cap H' = E^{M_{i+1}} \cap H^i \text{ and } J_{\nu_i}^{E^{M_i}} = J_{\nu_i}^{E^{M_{i+1}}}$$

QED(Lemma 3.4.1)

Using these facts we prove:

Lemma 3.4.2. *Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration. Let $h < i$. Then*

(a) $J_{\nu_h}^{E^{M_h}} = J_{\nu_h}^{E^{M_i}}$

(b) λ_h *is inaccessible in* M_i *and* $\nu_h = \lambda_h^+$ *in* M_i

(c) Let $h < j <_T i$. *Then* $\lambda_h \leq \text{crit}(\pi_{j,i}) < \lambda_i$.

(d) Let $h <_T i$. $\pi_{h,i}$ *is a total function on* M_h *iff* $[H, i]_T$ *is drop free.*

The proof is by induction on i. We leave the details to the reader.

Note. $h < i$ implies $\nu_h < \lambda_i$, since $\nu_h < \nu_i$ is a successor cardinal in M_i; hence $\nu_h \notin [\lambda_i, \nu_i)$.

Definition 3.4.3. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration.

- $lh(I)$ denotes the length of I
- If $\eta \leq lh(I)$ we set:

$$I|\eta =: \langle\langle M_i|i < \eta\rangle, \langle\nu_i|i + 1 < \eta\rangle, \langle\pi_{ij}|i \leq_T i < \eta\rangle, T \cap \eta^2\rangle.$$

Definition 3.4.4. Let $I = \langle\langle M_i\rangle, \ldots, T\rangle$ be a normal iteration of limit length η. By a *well founded cofinal branch* in I we mean a branch b in T such that

- $\sup b = \eta$
- b has at most finitely many truncation points
- Let $i \in b$ such that $b \setminus i$ is truncation free. Then

  $$\langle M_j|j \in b\rangle, \langle\pi_{hi}|i \leq h \leq j \text{ in } b\rangle$$

  has a well founded direct limit.

We leave it to the reader to prove:

Lemma 3.4.3. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration of limit length η. Let b be a well founded cofinal branch in I. I has a unique extension I' of length $\eta+1$ such that $I'|\eta = I$ and $T'''\{\lambda\} = b$. (Moreover, if $i \in b$ and $b \setminus i$ is drop free then:*

$$M'_\eta, \langle\pi'_{h,\eta}|h \in b \setminus i\rangle$$

is the transitivized direct limit of

$$\langle M_h|h \in b \setminus i\rangle, \langle\pi_{h,j}|h \in b \setminus i\rangle.$$

Note. We use Theorem 3.3.23 to show that M'_η is a premouse.

Note. It will be easier to talk about such limits if we have a notion of direct limit which can be applied to directed systems of *partial maps*. This could be defined quite generally, but the following version suffices for our purposes: Let $S = \langle S, <\rangle$ be a linear ordering. Let $\mathbb{A}_i$ be a model and let π_{ij} be a partial injection of $\mathbb{A}_i$ to $\mathbb{A}_j$ for $i \leq j$ in S. Assume that the maps commute (i.e. $\pi_{ij}\pi_{\kappa i} = \pi_{\kappa j}$) and that for sufficiently large $i \in S$ we have:

$$\pi_{ij} \text{ is a total map on } \mathbb{A}_8 \text{ for all } j \geq i \text{ in } I.$$

Let S' be the set of such i. We call:

$$\mathbb{A}, \langle \pi_i | i \in S \rangle$$

a direct limit of:

$$\langle \mathbb{A}_i | i \in S \rangle, \langle \pi_{ij} | i \leq j \text{ in } S \rangle$$

iff:

$$\mathbb{A}, \langle \pi_i | i \in S' \rangle$$

in a direct limit of:

$$\langle \mathbb{A}_i | i \in S' \rangle, \langle \pi_{ij} | i \leq j \text{ in } S' \rangle$$

and π_h is defined by: $\pi_h = \pi_i \pi_{hi}$ for $h \notin S'_1 i \in S$.

In §3.2 we defined $\mathbb{N}$ to be a Σ^**–ultrapower of* M *by* F with Σ^*–extension π (in symbols $\pi : M \to_F^* N$) iff F is close to M and $\pi : M \to_F^{(n)} N$ where $n \leq \omega$ is maximal such that $\text{crit}(F) < \rho_M^n$. Theorem 3.2.17 said that in this case π is Σ^*–preserving. We shall now show that in a normal iteration $E_{\nu_i}^{M_i}$ is always close to M_i^*. In order to utilize the full strength of this fact, we shall formulate it not only for normal iteration, but also for *potential* normal iteration in the following sense:

Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of length $i + 1$. If we attempt to extend I to an I' of length $i + 2$ by appointing the next ν_i, we call this attempt a *potential normal iteration*. The formal definition is:

Definition 3.4.5. A *potential normal iteration* of length $i + 2$ is a structure

$$\mathfrak{T}' = \langle \langle M_j | j \leq i \rangle, \langle \nu_j | j \leq i \rangle, \langle \pi_{ij} | i \leq j \leq i \rangle, T' \rangle$$

where:

- $I = \langle \langle M_j \rangle, \langle \nu_j | j < i \rangle, \langle \pi_{ij} \rangle, T \rangle$ is a normal iteration of length $i + 1$, where $T = T' \cap (i + 1)^2$
- $E_{\nu_i}^{M_i} \neq \emptyset$ and $\nu_i > \nu_j$ for $j < i$
- $hT'j \leftrightarrow (hTj \vee (h \leq_T \xi \wedge j = i))$ where:

$$\xi = T'(i + 1) =: \text{ the least } \xi \text{ such that } \kappa_i < \lambda_\xi.$$

If I' is a potential iteration and $\xi = T'(i + 1)$, we define $M_i^* = M_\xi U$ is in the usual way, (but we do not yet know whether M_i^* is extendable by $E_{\nu_i}^{M_i}$).

Note. (a)-(d) in the definition of *normal iteration* continue to hold. ((d) is trivial if $\xi = i$. If $\xi < i$, then $\tau_i < \lambda_\xi$ and $J^{E^{M_\xi}}_{\lambda_\xi} = J^{E^{M_i}}_{\lambda_\xi}$). But then M^*_i is defined and $\tau_i \in M^*_i$ is a cardinal in M^*_i. Let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n_{M^*_i}$. It is easily seen that, if the $\Sigma^{(n)}_0$ extension:

$$\pi' : M^*_i \longrightarrow^{(n)}_{E^{M_i}{}_{\nu_i}} M'$$

exists, we can turn I' into a normal iteration of length $i+2$ by setting:

$$M_{i+1} = M', \ \pi_{\xi,i+1} = \pi'$$

We now prove a basic fact about normal iteration:

Theorem 3.4.4. *Let I be a potential normal iteration of length $i+2$. Let $\xi = T(i+1)$. Then $E^{M_i}_{\nu_i}$ is close to M^*_i.*

Before proving this we note the obvious corollary:

Corollary 3.4.5. *Let I be a normal iteration. If $h = T(i+1)$ in I, then:*

$$\pi_{h,i+1} : M^*_i \to^*_{E_{\nu_i}} M_i.$$

Lemma 3.4.6. *Let I be a normal iteration. Let $h = T(i+1), i+1 \leq_T j$, where $(i+1, j]_T$ has no truncation point. Then:*

$$\pi_{h,j} : M^*_i \longrightarrow_{\Sigma^*} M_j \text{ strongly.}$$

In particular $\pi_{h,j}$" $P^n_{M^*_i} \subset P^n_{M_j}$ for $\rho^{n+1} = \rho^\omega$ in M^*_i.

Proof. By induction on j using Lemma 3.2.26, Lemma 3.2.27 and Lemma 3.2.28.

QED(Lemma 3.4.6)

We shall derive Theorem 3.4.4 from an even stronger statement:

Lemma 3.4.7. *Let I be a potential normal iteration of length $i+2$. Then*

$$\mathbb{P}(\tau_i) \cap \underset{\sim}{\Sigma}_1(M_i||\nu_i) \subset \underset{\sim}{\Sigma}_1(M^*_i).$$

We first show that Lemma 3.4.7 implies theorem 3.4.4. Since $F = E_{\nu_i}$ is weakly amenable, we need only show that $F_\alpha \in \underset{\sim}{\Sigma}_1(M^*_i)$ for $\alpha < \lambda_i$, where:

$$F_\alpha = \{x \subset \kappa_i | x \in M_i||\nu_i \wedge \alpha \in F(x)\}.$$

Let $k \in M_i||\nu_i$ map τ_i onto $J^E_{\tau_i}$. Then $k \in M^*_i$, since either $i = T(i+1)$ and $M^* \supset M_i||\nu_i$, or else $h = T(i+1) < i$, whence follows: $k \in J^{E^{M_i}}_{\lambda_h} = J^{E^{M^*_i}}_{\lambda_h} \subset M^*_i$. Set:

$$\tilde{F}_\alpha = \{\xi < \tau_i | k(\xi) \in F_\alpha\}.$$

Then $\tilde{F}_\alpha \subset \mathbb{P}(\tau_i)$ is $\underline{\Sigma}_1(M^*_i)$ by Lemma 3.4.7. Hence $F_\alpha = k''\tilde{F}_\alpha \in \underline{\Sigma}_1(M^*_i)$.

QED

We now prove Lemma 3.4.7. Suppose not. Let I be a counterexample of length $i+2$, where i is chosen minimally. Let $h = T(i+1)$. Then:

(1) $h < i$

Proof: Suppose not. Then $M^*_i = M_i||\mu$ where $\mu \geq \nu$. Hence $\underline{\Sigma}_1(M_i||\nu_i) \subset \underline{\Sigma}_1(M^*_i)$. Contradiction!

(2) $\nu_i = \mathrm{On}_{M_i}$ and $\rho^1_{M_i} \leq \tau_i$.

Proof: Suppose not. Let $A \subset \tau_i$ be $\underline{\Sigma}_1(M_i||\nu_i)$. Then $A \in \mathbb{P}(\tau_i) \wedge M_i \subset J^{E^{M_i}}_{\lambda_n}$, since $\lambda_h > \tau_i$ is inaccessible in ???. But $J^{E^{M_i}}_{\lambda_n} = J^{E^{M_i}}_{\lambda_n} \subset M^*_i$. Contradiction!

(3) i is not a limit ordinal.

Proof: Suppose not. Then $\sup\{\mathrm{crit}(\pi_{li})(\underset{T}{<} i\} = \sup_{l<i}\lambda_l$, so we can pick $L \underset{T}{<} i$ such that $\mathrm{crit}(\pi_{l,i}) > \lambda_h > \tau_i$ and $\pi_{l,i}$ is a total function on M_l. Then $\pi_{l,i} : M_l \to_{\Sigma_1} M_i$, where $M_i = \langle J^{E_i}_{\nu_i}, F\rangle$, where $F \neq \emptyset$. Hence $M_l = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle$ where $\overline{F} \neq \emptyset$. Let $A \subset \tau_i$ be $\underline{\Sigma}_1(M_i)$ such that $A \notin \underline{\Sigma}_1(M^*_i)$. We can assume l to be chosen large enough that $p \in \mathrm{rng}(\pi_{li})$, where A is $\Sigma_1(M_i)$ in the parameter p. Thus $A \in \underline{\Sigma}_1(M_l)$. Clearly $\overline{\nu} > \nu_j$ for all $j < l$, since $\nu_j \in M_l = \langle J^{\overline{F}}_{\overline{\nu}}, \overline{F}\rangle$.

Extend $I|l+1$ to a potential iteration I' of cf length $l+2$ by setting $\nu_l = \overline{\nu}$. Since $\mathrm{crit}(\pi_{l,i}) > I_i$, it follows easily that $\tau'_l = \tau_i, \kappa'_l = \kappa_i$, where τ_l, κ'_l are defined in the usual way. But then $M^*_i = (M'_l)^*$ and $A \in \underline{\Sigma}_1(M^*_i)$ by the minimality of i. Contradiction! QED (3)

Now let $i = j+1, \xi = T(i)$. Since $\pi_{\xi,i} : M^*_j \to_{\Sigma_1} M_i = \langle J^E_{\nu_i}, E_\nu\rangle$ where $E_{\nu_i} \neq \emptyset_i$ we have:

(4) $M^*_j = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}}\rangle$ where $\overline{E}_{\overline{\nu}} \neq \emptyset$.

(5) $\tau_i < \kappa_j$

Proof: $\tau_i < \lambda_j$ since $\tau_i = \kappa_i^{+M_i}$ and $\kappa_i < \lambda_h \leq \lambda_j$, where λ_j is inaccessible in M_i. But obviously $\kappa_i, \tau_i \in \mathrm{rng}(\pi_{\xi,i})$ by (4) where $[\kappa_j, \lambda_j) \cap \mathrm{rng}(\pi_{\xi i}) = \emptyset$. QED (5)

(6) $\pi_{\xi i} : M_j^* \to_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower.

Proof: Suppose not. Then $\kappa_j < \rho^1_{M_j^*}$. Hence $\pi_{\xi,i}$ is $\Sigma_0^{(1)}$–preserving. Hence $\pi_{\xi i}{}''\rho^1_{M_i^*} \subset \rho^1_{M_i}$. Hence $\tau_i = \pi_{\xi i}(\tau_j) < \rho^1_{M_i}$, contradicting (2).

QED (6)

But then:

(7) $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_j^*)$.

Proof: Let $A \subset \tau_i$ be $\Sigma_1(M_i)$ in the parameter p. Let $p = \pi_{\xi i}(f)(\alpha)$, where $f : \kappa_i \to M_i^*, f \in M_i^*$, and $\lambda < \lambda_j$. Then

$$A(\xi) \leftrightarrow \bigvee x A'(\zeta, x, p)$$

where A' is $\Sigma_0(M_i)$. Let $\overline{A}'$ be $\Sigma_0(M_j^*)$ by the same Σ_0 definition. Then, since $\pi_{\xi i}$ takes M_j^* cofinally to M_i by (6), we have

$$A(\zeta) \leftrightarrow \bigvee u \in M_j^* \bigvee x \in \pi_{\xi,i}(u) A'(\zeta, x, p).$$

By the minimality of i we know that $(E_{\nu_j})_\alpha \in \underline{\Sigma}_1(M_j^*)$ for $\alpha < \lambda_j$. But then:

$$A(\zeta) \leftrightarrow \bigvee u \in m_j^* \{\gamma < \kappa_j | \overline{A}'(\zeta, x, f(\gamma)\} \in (E_{\nu_i})_\alpha.$$

Hence A is $\underline{\Sigma}_1(M_j^*)$. QED (7)

Now extend $I|\xi + 1$ to a potential iteration I' of length $\xi + 2$ by setting $\nu'_\xi = \overline{\nu}$, where $M_j^* = M_\xi||\overline{\nu} = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}}\rangle$. Then $\kappa_i = \kappa'_\xi$ and $\tau_i = \tau'_\xi$, since $\pi_{\xi i} \restriction \kappa_j = \text{id}$. Hence $h = T(i+1) = T'(\xi+1)$ and $M_i^* = (M_\xi^*)'$. By the minimal choice of i we conclude

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_j^*) \subset \underline{\Sigma}_1(M_i^*).$$

Hence $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_(M_i^*)$ by (7). Contradiction! QED (Lemma 3.4.7)

3.4.3 Padded iterations

Normal iterations are often used to "compare" two premice M and M'. The *comparison iteration* or *coiteration* consists of a pair $\langle I, I'\rangle$ of iteration I of M and I' of M'. When we have reached M_i, M'_i, we proceed as follows: We look for the *least point of difference* — i.e. the least ν such that $M_i||\nu \neq M'_i||\nu$. Then $J_\nu^{E^{M_i}} = J_\nu^{E^{M'_i}}$ and $E_\nu^{M_i} \neq E_\nu^{M'_i}$. Then at least one of $E_\nu^{M_i}, E_\nu^{M'_i}$ is an extender. If both are extenders, we continue on the I–side with the index $\nu_i = \nu$. However, if, say, $E_\nu^{M_i}$ is an extender and $E_\nu^{M'_i} = \emptyset$, we iterate by $\nu_i = \nu$ on the I–side and on the I'–side do nothing. We then call i an *inactive point* on the I'–side and set: $M'_{i+1} = M'_i, \pi'_{i,i+1} = \text{id}$ with $i = T'(i+1)$ in I. Thus i is active on one or the other side and we have achieved: $M_{i+1}||\nu = M'_{i+1}||\nu = \emptyset$. (This is called "iterating away the least point of difference".) At a limit λ we choose on either side a well founded branch and continue with that.

If all goes well, we eventually reach a point i such that M_i, M'_i or one of $M_i = M'_i$ is a proper segment of the other.

In order to carry this out we need a slightly more flexible definition of "normal iteration", which admits inactive points. We therefore define:

Definition 3.4.6. A *padded normal iteration of length* μ is a sequence:

$$I = \langle\langle M_i | i < \mu\rangle, \langle \nu_i | i \in A\rangle, \langle \pi_{ij} | i \leq_T j\rangle, T\rangle$$

such that:

(1) $A \subset \{i : j+1 < \mu\}$ is called the set of *active points* in I.

(2) (a)-(h) of the previous definition hold, where (d)-(f) both require the assumption: $i \in A$.

(3) Let $h < j < \mu$ such that $[h, j) \cap A = \varnothing$. Then:

- $h \leq_T j, M_k = M_j, \pi_{hj} = \text{id}$.
- $i \leq h \longrightarrow (i \leq_T h \longleftrightarrow i <_T j)$ for $i < \mu$.
- $j \leq i \longrightarrow (j \leq_T i \longleftrightarrow h <_T i)$ for $i < \mu$. (In particular, if $i + 1 < \mu, i \notin A$, then $i = T(i+1), M_i = M_{i+1}$, and $\pi_{i,i+1} = \text{id}$.)

Note. This gives a new way of potentially extending I of length $i+1$. Instead of appointing ν_i, we could set: $i \notin A, M_{i+1} = M_i$.

All previous results go through a *mutatis mutandis*. We shall often use the term "normal iteration" so as to include padded normal iteration. We then call normal iterations in the sense of our previous definition *strict*. We can turn a padded iteration into a strict iteration simply by omitting the inactive points.

Conversely, we can turn a strict iteration into a padded iteration simply by inserting inactive points. The relevant lemmas are:

Lemma 3.4.8. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a (possibly padded) normal iteration of length μ. Let A be the set of active points in I. Set:*

$$A' =: \{i : i \in A \vee i+1 = \mu\}$$

Let $B \subset \mu$ such that $A' \subset B$. Let f be the monotone enumeration of B. Then:

$$I' = \langle\langle M_{f(i)}\rangle, \langle\nu_{f(i)}\rangle, \langle\pi_{f(i),f(j)}\rangle, T'\rangle$$

is a normal iteration , where $T' = \{\langle i,j\rangle : f(i)Tf(j)\}$. (Moreover I' is strict if $B = A'$).

Proof. (a)-(i) are satisfied by I'.

Conversely:

Lemma 3.4.9. *Let I, μ be as above. Let $f : \mu \longrightarrow \mu'$ be monotone such that* $\operatorname{lub} f"\mu = \mu'$ *if μ is a limit ordinal. Set: $\overline{f}(i) = \operatorname{lub} f"i$ for $i < \mu$. For $i < \mu'$ set:*

$\xi_i =$ *that ξ such that either $\overline{f}(\xi) \leq i \leq f(\xi)$, or else $\xi + 1 = \mu$ and $f(\xi) < i$.*

Define:

$$I' = \langle\langle M'_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$$

by:

$$M'_i = M_{\xi_i}, \pi'_{ij} = \pi_{\xi_i,\xi_j}, T' = \{\langle i,j\rangle : \xi_i T \xi_j\}$$

and:

$$\nu'_i = \begin{cases} \nu_{\xi_i} & \text{if } i = f(\xi_i) \\ \text{otherwise undefined} & \end{cases}$$

Then I' is a normal iteration.

Proof: I' satisfies (a)-(i).

Note. Lemma 3.4.9 enables to recover I form the I' in Lemma 3.4.8.

We leave the proof to the reader.

3.4.4 n–iteration

In a normal iteration we always take Σ^* ultrapowers. For technical reasons, however, we may sometimes want to bound the degree of preservation of our ultraproducts. In a 0–*iteration* for instance, we would use the ordinary Σ_0

ultrapower to pass from M_i to M_{i+1}, as long as no $h \leq_T i+1$ is a truncation point. If, on the other hand, we have reached a truncation point $h \leq_T i+1$, we then revert to the full Σ^*–ultrapowers. More generally:

Definition 3.4.7. Let $n \leq \omega$. By a *normal n–iteration* of M of *length* μ we mean

$$\langle\langle M_i | i < \mu\rangle, \langle \nu_i | i+1 < \mu\rangle, \langle \pi_{ij} | i \leq_T \rangle, T\rangle,$$

where (a) – (e) and (g) ,(h) in the definition of "normal iteration" hold, and in addition:

(f) Let $h = T(i+1)$. If τ_i is a cardinal in M_h and π_{jh} is a total map on M_j for jTh, then $\pi_{h,i+1} : M_h \to^{(m)}_{E_{\nu_i}} M_{i+1}$, where $m \leq n$ is maximal such that $\kappa_i < \rho^m_{M_h}$.

Otherwise $\pi_{h,i+1} : M^*_i \to^{(m)}_{E_{\nu_i}} M_{i+1}$, where M^*_i is defined as before and $m \leq \omega$ is maximal such that $\kappa_i < \rho^m_{M^*_i}$.

Note. An ω–iteration is then the same as a normal iteration n the sense of our previous definition. We also call such iterations $*$*–iterations*, since we then always take the Σ^* ultrapowers. $*$–iterations are the ones we are interested in.

It is easily seen that the conclusions of Lemma 3.4.2 hold for normal n–iterations. Lemma 3.4.3 also holds for these iterations and Lemma 3.4.7 holds *mutatis mutandis*. We leave this to the reader. More suprising is:

Theorem 3.4.10. *Theoem 3.4.4 holds for normal n–iterations.*

Before proving this, we again note some consequences. It follows easily that:

Corollary 3.4.11. *Let I be a normal n–iteration. Let $h = T(i+1)$. Let m be maxiomal such that $\kappa_i < \rho^m_{M^*_i}$. Assume either that $m \leq n$ or that there is a $j \leq_T i+1$ which is a drop point. Then:*

$$\pi_{h,i+1} : M^*_i \to^*_{E_{\nu_i}} M_{i+1}.$$

In all other cases we have:

$$\pi_{h,i+1} : M^*_i \to^{(n)}_{E_{\nu_i}} M_{i+1}.$$

But then by induction on i we get:

Corollary 3.4.12. *Let I be as above. Let π_{ij} be a total map on M_i. If there is a drop point j such that jTi, then π_{ij} is Σ^*–preserving. Otherwise it is $\Sigma^{(n)}_0$–preserving.*

As before, we derive Lemma 3.4.10 from:

Lemma 3.4.13. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a potential n–iteration of length $i+2$. Then $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_i(M_i||\nu_i) \subset \underline{\Sigma}_1(M_i^*)$.*

The derivation of Lemma 3.4.10 from Lemma 3.4.13 is exactly as before. We prove Lemma 3.4.13. Almost all steps in the proof of Lemma 3.4.7 go through as before. The only difficulty occurs in the proof of (6), where we derived that $\pi_{\xi,i}$ is $\Sigma_0^{(1)}$–preserving from: $\kappa_j < \rho^1_{M_j^*}$. If $n \geq 1$, this is unproblematical. Now assume $n = 0$. If there is a drop point $l \leq_T i$, then $\pi_{\xi,i}$ is Σ^*–preserving and there is nothing to prove. Now suppose there is no such drop point.

By the definition of "0–iteration" we then have: $\pi_{\xi,i} : M_j^* \to^0_{E_{\nu_j}} M_i$, which was to be proven.

All other steps in the proof go through. QED (Lemma 3.4.13)

This proves Theorem 3.4.10.

The concept "padded n–iteration" is defined exactly as before. As before, every padded iteration can be converted into a strict iteration by omitting the inactive points, and every strict iteration can be expanded to a padded iteration by inserting inactive points. We leave this to the reader.

3.4.5 Copying an iteration

Suppose that I is a normal iteration of a premouse M and $\sigma : M \to_{\Sigma^*} M'$, where M' is a premouse. We can attempt to "copy" I onto an iteration I' of M' by repeating the same steps modulo σ. We define:

Definition 3.4.8. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a strict normal iteration of M. Let $\sigma : M \to_{\Sigma^*} M$, where M' is a premouse. We call $I' = \langle\langle M_i'\rangle, \langle\nu_i'\rangle, \langle\pi_{ij}'\rangle, T'\rangle$ a *copy of I induced by* $\langle\sigma, M'\rangle$ with *copying map* $\langle\sigma_i | i < lh(I)\rangle$ iff the following hold:

(a) $lh(I') = lh(I)$ and $T' = T$

(b) $\sigma_i : M_i \to_{\Sigma^*} M_i'$ and $\sigma_0 = \sigma$

(c) $\sigma_i\pi_{li} = \pi'_{li}\sigma_j$ for $l \leq_T i$

(d) $\sigma_i \restriction \lambda_l = \sigma_l \restriction \lambda_l$ for $l \leq i$

(e) $\nu'_i = \sigma_i(\nu_i)$ for $\nu_i \in M_i$. Otherwise $\nu'_i = \mathrm{On} \cap M'_i$.

Note. This definition can easily be extended to padded normal iterations. (b) – (e) are then stipulated for active points, and for inactive points we stipulate:

(f) If i is inactive in I, it is inactive in I' and $\sigma_{i+1} = \sigma_i$.

We shall often formulate our definitions and theorems for strict iteration, leaving it to the reader to discover — mutatis mutandis — the correct version for padded iterations. In particular, the remaining theorems in this section will assume strictness.

We also define:

Definition 3.4.9. $\langle I, I', \langle \sigma_i | i < lh(I) \rangle \rangle$ is a *duplication* iff I, I' are normal iterations and I' is a copy of I with copying maps $\langle \sigma_i \rangle$.

Lemma 3.4.14. *Let I' be a copy of I with copying maps $\langle \sigma_i \rangle$. Let $h = T(i+1)$.*

(i) If $i+1$ is a drop point in I, then it is a drop point in I' and $M'^*_i = \sigma_h(M^*_i)$.

(ii) If $i+1$ is not a drop point in I, it is not a drop point in I'. (Hence $M^*_i = M_h, M'^*_i = M'_h$.)

(iii) Let $F = E^{M_i}_{\nu_i}, F' = E^{M'_i}_{\nu'_i}$. Then:

$$\langle \sigma_h \restriction M^*_i, \sigma_i \restriction \lambda_i \rangle : \langle M^*_i, F \rangle \to \langle M'^*_i, F' \rangle$$

as defined in §3.2.

(iv) $\sigma_{i+1}(\pi_{h,i+1}(f)(\alpha)) = \pi'_{h,i+1}\sigma_h(f)(\sigma_i(\alpha))$ for $f \in \Gamma^*(\kappa_i, M^*_i)\alpha < \lambda_i$.

(v) $\sigma_j(\nu_i) = \nu'_i$ for $i < j$.

Proof:

(i) Let $h = T(i+1)$. Then $M^*_i = M_h||\mu$, where $\mu \in M_h$ is maximal such that τ_i is a cardinal in $M_h||\mu$. But $\tau'_i = \sigma_i(\tau_i) = \sigma_h(\tau_i)$ by (d), (e). Hence $\sigma_h(\mu) = \mu'$, where μ' is maximal such that τ'_i is a cardinal in M'_h, and $\sigma_h(M_h||\mu) = M'_h||\mu'$.

(ii) If τ is a cardinal in M_h, then $\tau_i' = \tau_h(\tau)$ is a cardinal in M_h', since σ_h is Σ_1–preserving.

(iii) Clearly $\sigma_h \restriction M_i^* : M_i^* \to_{\Sigma^*} M_i'^*$ by (i) and (ii). Now let $x \in \mathbb{P}(\kappa_i) \cap M_i^*$ and $\alpha_1, \ldots, \alpha_n < \lambda_0$. Since $\sigma_i : M_i \longrightarrow M_i'$ is Σ^*-preserving we have:

$$\langle \vec{\alpha} \rangle \in F(x) \leftrightarrow \langle \sigma_i(\vec{\alpha}) \rangle \in F'(\sigma_i(x)).$$

But $\sigma_i(x) = \sigma_h(x)$, since by (d) we have: $\sigma_i \restriction J_{\lambda_n}^{E^{M_i}} = \sigma_h \restriction J_{\lambda_h}^{E^{M_h}}$.

(iv) If $f \in M_i^*$, then by (c):

$$\sigma_{i+1}\pi_{h,i+1}(f) = \pi'_{h,i+1}\sigma_h(f).$$

Otherwise $f(\xi) \simeq G(\xi, q)$ where $q \in M_i^*$ and G is a good $\Sigma_1^{(n)}(M_i^*)$ function for an n such that $\kappa_i < \rho_{M_i^*}^{n+1}$. But then:

$$\begin{aligned} \sigma_{i+1}\pi_{h,i+1}(f)(\xi) &\simeq G'(\xi, \sigma_{i+1}\pi_{h,i+1}(q)) \\ &\simeq G'(\xi, \pi'_{h,i+1}\sigma_h(q)) \\ &\simeq \pi'_{h,i+1}\sigma_h(f) \end{aligned}$$

where G' is $\Sigma_1^{(n)}(M_i'^*)$ by the same good definition.

(v) If $j > i+1$, then $\nu_i < \lambda_{i+1}$ and $\sigma_j(\nu_i) = \sigma_{i+1}(\nu_i)$. But letting $h = T(i+1)$, we have:

$$\sigma_{i+1}(\nu_i) = \sigma_{i+1}\pi_{h,i+1}(\tau_i) = \pi'_{h,i+1}\sigma_h(\tau_i),$$

where

$$\sigma_h(\tau_i) = \sigma_i(\tau_i) = \tau_i', \text{ since } \tau_i < \lambda_h.$$

Hence $\sigma_{i+1}(\nu_i) = \pi'_{h,i+1}(\tau_h') = \nu_i'$.

QED (Lemma 3.4.14)

It is apparent from Lemma 3.4.14 that there is only one way to extend a copy of $I|i+1$ to a copy of $I|i+2$. Moreover, the copying map σ_i is unique. Similarly, if η is a limit ordinal and I' is a copy of $I|\mu$ with copying maps $\langle \sigma_i | i < \eta \rangle$, ther is only one way to extend I' to a copy of $I|\eta+1$, for then:

$$M', \langle \pi'_{i,\eta} | iT\eta \rangle$$

is the direct limit of:

$$\langle M_i' | i < \eta \rangle, \langle \pi'_{ij} | i \leq_T j <_T \eta \rangle,$$

and σ_η is defined by:

$$\sigma_\eta \pi_{i\eta} = \pi'_{i\eta}\sigma_i \text{ for } i <_T \eta.$$

Hence, by induction on $lh(I)$ we get:

Lemma 3.4.15. *Let I be a normal iteration of M. Let $\sigma : M \to_{\Sigma^*} M'$. Then there is at most one copy I' of I induced by σ. Moreover, the copying maps σ_i are unique.*

Now suppose that I is a normal iteration of length $i+1$ and I' is a copy of I with copying maps $\langle \sigma_h \mid h \leq i\rangle$. Extend I to a potential iteration $\tilde{I}$ of length $i+2$ by appointing ν_i. Extend I' to a potential iteration $\tilde{I}'$ by appointing:

$$\nu_i' = \begin{cases} \sigma_i(\nu_i) \text{ if } \nu_i \in M_i \\ \mathrm{On} \cap M_i' \text{ if } \nu_i = \mathrm{On} \cap M_i. \end{cases}$$

We call $\langle \tilde{I}, \tilde{I}', \langle \sigma_j || \leq i\rangle\rangle$ a *potential duplication* of length $i+2$. The formal definition is:

Definition 3.4.10. Let I, I' be potential iteration of length $i{+}2$. $\langle \tilde{I}, \tilde{I}', \langle \sigma_j | j \leq i\rangle\rangle$ is a *potential duplication* of length $i+2$ iff

- $\langle \overline{I}, \overline{I}', \langle \sigma_j | j \leq i\rangle\rangle$ is a duplication of length $i+1$, where $\overline{I} = \tilde{I}|i+1, \overline{I}' = I'|i+1$.
- $\sigma_i(\nu_i) = \nu_i'$ if $\nu_i \in M_i$. Otherwise $\nu_i' = \mathrm{On} \wedge M_i'$.

Note. It is then easily seen that $T(i+1) = T'(i+1)$. We also know that $E^{M_i}_{\nu_i}$ is close to M_i^n and $E^{M_i'}_{\nu_i'}$ is clost to M_i'. The following theorem is an analogue of theorem 3.4.7

Lemma 3.4.16. *Let $\langle I, I', \langle \sigma_i\rangle\rangle$ be a potential duplication of length $i+2$. Let $h = T(i+1)$. Then:*

$$\langle \sigma_h \restriction M_i^*, \sigma_i \restriction \lambda_i\rangle : \langle M_i^*, F\rangle \to^* \langle {M'}_i^*, F'\rangle$$

(as defined in §3.2) where $F = E^{M_i}_{\nu_i}, F' = E^{M_i'}_{\nu_i'}$.

Before proving the theorem, we note some of its consequences. It gives us exact criteria for determining whether the copying process can be continued one step further.

Lemma 3.4.17. *Let I be a normal iteration of M of length $i+2$. Let $\sigma : M \to M'$ induce a copy I' of $I|0+1$ with copying maps $\langle \sigma_j | j \leq i\rangle$. Set:*

$$\nu_i' = \begin{cases} \sigma_i(\nu_i) \textit{ if } \nu_i \in M_i \\ \mathrm{On} \cap M_0' \textit{ if } \nu_i = \mathrm{On} \cap M_i \end{cases}$$

Then σ induces a copy of I iff ${M'}_i^*$ is Σ^*–extendible by $E^{M_i'}_{\nu_i'}$.

Proof: If ${M'}^*_i$ is not extendible, then no such copy can exist. Now let ${M'}^*_i$ be extendible. Let $\pi'_{h,i+1} : {M'}^*_i \to^*_{E^{{M'}^*_i}_{\nu'_i}} {M'}^*_{i+1}$. By theorem 3.4.16 and Lemma 3.2.23 it follows that there is a unique $\sigma : M_{i+1} \to_{\Sigma^*} M'_{i+1}$ such that $\sigma\pi_{h,i+1} = \pi'_{h,i+1} \cdot \langle \sigma_h \restriction M^*_i \rangle$, where $h = T(i+1)$. Set: $\sigma_{i+1} =: \sigma$. This gives us the copy I'' of I with copying maps $\langle \sigma_j | j \leq 0+1 \rangle$.

QED (Lemma 3.4.17)

We also have:

Lemma 3.4.18. *Let I be a normal iteration of M of length $\eta+1$, where η is a limit ordinal. Let $\sigma : M \to_{E^*} M'$ induce a copy I' of $I|\eta$. We can extend I' to a copy of I induced by σ iff $b = T''\{\eta\}$ is a well founded branch in I'.*

The proof is left to the reader.

We also note:

Lemma 3.4.19. *Let I be a normal iteration of limit length. Let I' be a copy of I. If b is a cofinal well founded branch in I', then it is a cofinal well founded branch in I.*

The proof is left to the reader.

We now turn to the proof of theorem 3.4.16. As with theorem 3.4.7 we derive it from an even stronger lemma:

Lemma 3.4.20. *Let $\langle I, I', \langle \sigma_i \rangle \rangle$ be a potential duplication of length $i+2$. Let $A \subset \tau_i$ be $\Sigma_1(M_i||\nu_i)$ in a parameter p. Let $A' \subset \tau'_i$ be $\Sigma_1(M_i||\nu_i)$ in $\sigma_i(p)$ by the same definition. Then A is $\Sigma_1(M^*_i)$ in a parameter q and A' is $\Sigma_1({M'}^*_i)$ in $\sigma_h(q)$ by the same definition, where $h = T(i+1)$.*

The derivation of theorem 3.4.16 from lemma 3.4.20 is a virtual repetition of the proof of theorem 3.4.4 from lemma 3.4.7. We leave it to the reader.

Lemma 3.4.20 is proven by a virtual repetition of the proof of lemma 3.4.7, making changes as necessary. We give a brief sketch of the proof:

Suppose not. Let I, I', ν_i, ν'_i be counterexamples of length $i+1$, where i is chosen minimally. Let $h = T(i+1) = T'(i+1)$. Then:

(1) $h < i$.
Suppose not. Then $M_i||\nu_i \subset M^*_i$ and $M'_i||\nu'_i \subset {M'}^*_i$ as before. If $\nu_i \in M^*_i$, then $\sigma_i(M||\nu_i) = M'_i||\nu'_i$. Hence $A \in M^*_i$ and $\sigma_i(A) = A'$. Contradiction!

(2) $\nu_i = \text{On}_{M_i}$ and $\rho^i_{M_i} \leq \tau_i$.
Otherwise, as before $A \in \mathbb{P}(\tau_i) \cap M_i^*, A' \in \mathbb{P}(\tau_i) \cap M'^*_i$ and $\sigma_h(A) = \sigma_i(A) = A'$. Contradiction!

(3) i is not a limit cardinal.
The proof of this is a virtual repetition of the argument given in the proof of lemma 3.4.7. We leave it to the reader.

Now let $i = j+1, \xi = T(i)$. Exactly as before we have:

(4) $M_j^* = \langle J_\nu^E, E_\nu \rangle, M'^*_j = \langle J_{\nu'}^{E'}, E'_{\nu'} \rangle$ where $E_\nu, E'_\nu \neq \emptyset$.

(5) $\tau_i < \kappa_j$.

(6) $\pi_{\xi,i} : M_j^* \to_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower (and therefore cofinal). Similarly for $\pi'_{\xi,i} : M'^*_j \to_{E_{\nu'_j}} M'_i$. By the minimality of σ we know that for all $\alpha < \lambda_j$, $(E^{M_j}{}_j)_\alpha$ is $\Sigma_1(M_j^*)$ in a parameter r and $(E^{M_i}{}_{\nu'_i})_{\sigma_i(\alpha)}$ is $\Sigma_1(M'^*_j)$ in $\sigma_\xi(r)$ by the same definition. Using this we can repeat the argument in the proof of Lemma 3.4.7 to get:

(7) A is $\Sigma_1(M_j^*)$ in a q and A' is $\Sigma_1(M'^*_j)$ in $\sigma_\xi(q)$ by the same definition.

Now extend $I|\xi+1$ to a potential iteration $\tilde{I}$ of length $\xi+2$ by setting $\tilde{\nu}_\xi = \nu$, where ν is as in (4). Extend $I'|\xi+1$ to $\tilde{I}'$ by setting $\tilde{\nu}_\xi = \nu'$ where ν' is as in (4). Then $\kappa_i = \tilde{\kappa}_\xi, \tau_i = \tilde{\tau}_\xi, \kappa'_i = \tilde{\kappa}_\xi, \tau'_i = \tilde{\tau}'_\xi$ as before. Hence $h = \tilde{T}(\xi+1) = \tilde{T}'(\xi+1)$ and $M_i^* = \tilde{M}_\xi^*, M'^*_i = \tilde{M}'^*_\xi$. By this minimality of i we conclude that A is $\Sigma_1(M_i^*)$ ia a q and A' is $\Sigma_1(M'^*_i)$ in $\sigma_h(q)$ by the same definition. Contradiction! QED (Lemma 3.4.20)

3.4.6 Copying an n–iteration

Definition 3.4.11. Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal n–iteration ($n \leq \omega$). Let $\sigma : M \to_{\Sigma_1^M} M'$. We call:

$$I' = \langle \langle M'_i \rangle, \langle \nu'_i \rangle, \langle \pi'_{ij} \rangle, T' \rangle$$

a *copy* (or *n–copy*) of I *induced by* $\langle \sigma, M' \rangle$ iff I' is an n–iteration satisfying (a), (c), (d), (e) of the previous definition together with

(b') $\sigma_0 = \sigma$ and $\sigma : M_i \to_{\Sigma_1^{(n)}} M'_i$. Moreover, if some $h \leq_T i$ is a truncation point, then σ_i is Σ^*–preserving.

The notion "n–duplication" and "potential n–duplication" are defined as before. Lemma 3.4.14 goes through as before exept (iv) must be reformulated as:

(iv') If no $l \leq_T i+1$ is a truncation point and $\kappa_i < \rho^n_{M_h}$, then:

$$\sigma_{i+1}(\pi_{h,i+1}(f))(\alpha) = \pi'_{h,i+1}\sigma_i(f)(\sigma_i(\alpha))$$

for $f \in \Gamma^n_*(\kappa_i, M_h), \alpha < \lambda_i$. In all other cases the equation holds for

$$f \in \Gamma^*(\kappa_i, M^*_i), \alpha < \lambda_i.$$

Lemma 3.4.15 then holds as before. Theorem 3.4.16 and lemma 3.4.17 – 3.4.19 then go through as before. By theorem 3.4.16 we also get:

Lemma 3.4.21. *Let $\langle I, I', \langle\sigma_i\rangle\rangle$ be an n–duplication. Let $i <_T j$ in I such that π_{ij} is total on M_i.*

(a) If no $l \leq_T i$ is a truncation point and $\kappa_i < \rho^n_{M_i}$, then $\pi_{ij} : M_i \to_{\Sigma^{(n)}_1} M_j$.

(b) In all other cases π_{ij} is Σ^*–preserving.

These lemmas and theorems hold *mutatis mutandis* for padded n–iterations. The details are left to the reader.

3.5 Iterability

A mouse is a premouse which is iterable. Iterability is, however, as complex a notion as that of iterating itself. We begin with *normal iterability* which says that any normal iteration of M constructed accordig to an appropriate strategy, can be continued.

3.5.1 Normal iterability

Definition 3.5.1. A premouse M has the *normal uniqueness property* (NUP) iff every normal iteration of M of limit length has at most one cofinal well founded branch. The simplest mice, such as $0^{\#}, 0^{\#\#}$ etc. are easily seen to have this property. Unfortunately, however, there are mice which do not. If a premouse M does satisfy NUP, then normal iterability can be defined by: