The notion "n–duplication" and "potential n–duplication" are defined as before. Lemma 3.4.14 goes through as before exept (iv) must be reformulated as:

(iv') If no $l \leq_T i+1$ is a truncation point and $\kappa_i < \rho^n_{M_h}$, then:

$$\sigma_{i+1}(\pi_{h,i+1}(f))(\alpha) = \pi'_{h,i+1}\sigma_i(f)(\sigma_i(\alpha))$$

for $f \in \Gamma^n_*(\kappa_i, M_h), \alpha < \lambda_i$. In all other cases the equation holds for

$$f \in \Gamma^*(\kappa_i, M^*_i), \alpha < \lambda_i.$$

Lemma 3.4.15 then holds as before. Theorem 3.4.16 and lemma 3.4.17 – 3.4.19 then go through as before. By theorem 3.4.16 we also get:

Lemma 3.4.21. *Let $\langle I, I', \langle \sigma_i \rangle \rangle$ be an n–duplication. Let $i <_T j$ in I such that π_{ij} is total on M_i.*

(a) If no $l \leq_T i$ is a truncation point and $\kappa_i < \rho^n_{M_i}$, then $\pi_{ij} : M_i \to_{\Sigma^{(n)}_1} M_j$.

(b) In all other cases π_{ij} is Σ^*–preserving.

These lemmas and theorems hold *mutatis mutandis* for padded n–iterations. The details are left to the reader.

3.5 Iterability

A mouse is a premouse which is iterable. Iterability is, however, as complex a notion as that of iterating itself. We begin with *normal iterability* which says that any normal iteration of M constructed accordig to an appropriate strategy, can be continued.

3.5.1 Normal iterability

Definition 3.5.1. A premouse M has the *normal uniqueness property* (NUP) iff every normal iteration of M of limit length has at most one cofinal well founded branch. The simplest mice, such as $0^{\#}, 0^{\#\#}$ etc. are easily seen to have this property. Unfortunately, however, there are mice which do not. If a premouse M does satisfy NUP, then normal iterability can be defined by:

Definition 3.5.2. Let M satisfy NUP, M is *normally iterable* iff every normal iteration of M can be continued — i.e.

- If I is a normal iteration of M of limit length, then it has a cofinal well founded branch.
- If I is a potential iteration of length $i+2$, then M_i^* is $*$–extendible by $E_{\nu_i}^{M_i}$.

If M does not satisfy NUP, we say that it is normally iterable if there exists a *strategy* for picking cofinal well founded branches such that any iteration executed in accordance with that strategy could be continued. We first define:

Definition 3.5.3. A *normal iteration strategy* is a partial function S on normal iterations of limit length such that $S(I)$, if defined, is a well founded cofinal branch in I. We call it a strategy *for M* if its domain is restricted to iterations of M.

Definition 3.5.4. A normal iteration $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle x_{ij}, T\rangle$ *conforms* to the iteration strategy S iff, whenever, $\eta < \operatorname{lh} I$ is a limit ordinal, then $T''\{\eta\} = S(I|\eta)$.

Definition 3.5.5. A normal iteration strategy S is *α–successful* for a premouse M iff every S–conforming iteration of M of length $< \alpha$ can be continued in an S–conforming way. In other words:

- If I is of limit length $< \alpha$, then $S(I)$ is defined
- If I is a potential normal iteration length $i+2 < \alpha$, then M_i^* is $*$–extendible by $E_{\nu_i}^{M_i}$.

Definition 3.5.6. M is *normally α–iterable* iff there exists an α–successful strategy for M.

Definition 3.5.7. M is *normally iterable* iff it is normally α–iterable for all α.

Note. It might seem more natural to take "normal iterable" as meaning that M is ∞–iterable, but that is a second order property, which we cannot express in ZFC.

Note. If M has NUP, then any two iteration strategies for M must coincide on their common domain. Hence, in this case, our initial definition of "normally iterable" is equivalent to the definition just given. It is then also equivalent to the second order statement that M is ∞–iterable.

Definition 3.5.8. M is *uniquely normally iterable* iff it is normally iterable and satisfies NUP.

Proving iterability is a central problem of inner model theory. There are large classes of premice for which it is unsolved. The success we have had to date depends strongly on NUP. Whenever we have been able to prove the iterability M, it is either because M satisfies NUP, or because we derive its iterability from that of another premouse which satisfies NUP.

Note. In the above definition we take "normal iteration" as meaning "padded normal iteration". One can, of course, define *strict iteration strategy, strictly α–successful* and *strictly α–iterable* in the obvious way. But in fact every strictly α–iterable premouse is α–iterable, since every strictly successful strategy S can be expanded to an α–successful S^* as follows. Let:

$$I = \langle\langle M_i\rangle, \langle\nu_i | i \in A\rangle, \langle\pi_{ij}, T\rangle$$

be padded iteration of limit length η. If A is cofinal in η, let $\langle\alpha_i | i < \mu\rangle$ be the monotone enumeration of A and set:

$$I' = \langle\langle M_{\alpha_i}\rangle, \langle\nu_{\alpha_0}\rangle, \langle\pi_{\alpha_i,\alpha_j}\rangle, \{\langle i, j\rangle | \alpha_i T \alpha_j\}\rangle.$$

Then I' is strict and we set:

$$S^*(\mathbb{I}) \simeq \{i | \bigvee j \in S(\mathbb{I}') i T_{\alpha_j}\}.$$

If A is not cofinal in η, let $j < \eta$ such that $[j, \eta] \cap A = \emptyset$. $S^*(I)$ is then defined to be the unique cofinal well founded branch:

$$\{i | i T_j \vee j \leq i < \eta\}.$$

3.5.2 The comparison iteration

As mentioned earlier, we can "compare" two normally iterable premice via a pair of padded normal iterations known as the *coiteration* or *comparison iteration*. We define:

Definition 3.5.9. Let M, N be premice. M is a *segment of* N (in symbols: $M \lhd N$) iff $M = N||\eta$ for an $\eta \leq \mathrm{On}_N$.

If neither of M^0, M^1 is a segment of the other, there is a *first point of difference* ν_0 defined as the least ν such that $M^0||\nu \neq M^1||\nu$. Then $J_{\nu_0}^{E^{M^0}} = J_{\nu_0}^{E^{M^1}}$ and $E^{M_0}{}_{\nu_0} \neq E^{M'}{}_{\nu_0}$. Set : $\pi_{0,1}^h : M^h \longrightarrow_{E_{\nu_0}} M_1^h$ if $E^{M^h}{}_{\nu_0} \neq \varnothing$. Otherwise set: $M_1^h = M^h$, $\pi_{0,1}^h = \mathrm{id}$. Then $M_1^0||\nu_0 = M_1^*||\nu_0$. If M_1^0, M_1^1

have a point ν_1 of difference, then $\nu_1 > \nu_0$ and we can repeat the process to get M_2^h etc. Suppose that $\mathrm{card}(M^h) < \Theta$ for $h = 0, 1$ where $\Theta + 1$ is regular and each M^h is $\Theta + 1$ iterable. The comparison process then continues until we have a pair of iterations of length $i + 1$, where either $i = \Theta$ of $i < \Theta$ and M_i^0, M_i^1 have no point of difference. (Hence one is a segment of the other.) Using the initial segment condition we shall show that the comparison must terminate at an $i + 1 < \Theta$. The formal definition is:

Definition 3.5.10. Let θ be a regular cardinal. Let M^0, M^1 be premice of cardinality $< \theta$ which are normally $E^+_{\mathrm{top}}1$-iterable. Let S^n be a successful $E^+_{\mathrm{top}}1$ strategy for M^n ($n = 0, 1$). The *coiteration* of M^0, M^1 given by $\langle S^0, S^1\rangle$ is a pair $\langle I^0, I^1\rangle$ of padded normal iteraions of common length $\mu+1 \leq \theta + 1$ with coindices $\langle \nu_i | i < \mu\rangle$ such that:

$$I^n = \langle\langle M_i^n\rangle, \langle \nu_i | i \in A^n\rangle, \langle \pi^n_{i,j}\rangle, T^n\rangle$$

and:

- $M_0^n = M^n$
- If M_i^0, M_i^1 are given and $i < \theta$, then:

  $$\nu_i \cong \text{ the first point of difference } \nu \text{ such that } M_i^0||\nu \neq M_i^1||\nu$$

- If $i = \theta$ of ν_i does not exist, then $i = \mu$.

There is obviously at most one coiteration $\langle I^0, I^1\rangle$. To see this, suppose $\langle I'^0, I'^1\rangle$ to be a second one and prove: $I^n|j + 1 \cong I^{(n)}|j + 1$ for $n = 01$ by induction on $j \leq \mu$ (we take $I^n|j = I^n$ if $\mathrm{lh}(I^n) \leq j$). We leave this to the reader. Finally, by induction on $j \leq \mu$ we prove the existence of $\langle I^0|i+1, I^1|j+1\rangle$ satisfying the above condition for $j \leq i$. This we also leave to the reader.

We now prove the *Comparison Lemma*:

Lemma 3.5.1. *The coiteration terminates before* Θ.

Proof: Suppose not. Since $\mathrm{card}(M^h) < \Theta$, it follows easily that:

(1) $I^h \in H_{\Theta^+}$ for $h = 0, 1$

Set $Q = H_{\Theta^+}$. By a Löwenheim–Skolem argument there is $X \prec Q$ such that $\mathrm{card}(X) < \Theta$, $X \cap \Theta$ is transitive, and $I^0, I^1 \in X$. Let:

(2) $\sigma : \overline{Q} \overset{\sim}{\longleftrightarrow} X$, where $\overline{Q}$ is transitive.

Then $\sigma : \overline{Q} \prec Q$. Let $\sigma(\overline{\mathbb{I}}^h) = \mathbb{I}^h (h = 0, 1)$. It is easily seen that:

(3)
- $\overline{\Theta} =: \Theta \cap X$ is the critical point of σ
- $\sigma(\overline{\Theta}) = \Theta$
- $\sigma \upharpoonright \overline{H} = \mathrm{id}$, where $\sigma(\overline{H}) = H_\Theta$.

But then $\overline{\Theta}$ is a limit point of the club set $T^{h\prime\prime}\{\Theta\}$ for $h = 0, 1$. Hence:

(4) $\overline{\Theta} <_{T^h} \Theta (h = 0, 1)$.

Let $\sigma(\overline{I}^h) = I^h$, where:

$$\overline{I}^h = \langle \langle \overline{M}_i^h \rangle, \langle \overline{\nu}_i^h \rangle, \langle \overline{\pi}_{ij}^h \rangle, \overline{T}^h \rangle.$$

Then:

(5) $\overline{I}^n = I^h | \overline{\Theta} + 1$.

Proof: $\overline{I}^n | \overline{\Theta} = I^h | \overline{\Theta}$ follows by (3). But then by (4):

$$i \overline{T}^h \overline{\Theta} \leftrightarrow i T^h \Theta \leftrightarrow i T^h \Theta \text{ for } i < \overline{\Theta}.$$

But then $\langle \overline{M}_{\overline{\Theta}}^h \rangle, \langle \overline{\pi}_{i\overline{\Theta}}^h | i \overline{T}^h \overline{\Theta} \rangle$ is the direct limit of:

$$\langle M_i | i < \Theta \rangle, \langle \pi_{ij} | i \leq_{T^h} <_{T^n} \Theta \rangle.$$

Hence $\overline{M}_{\overline{\Theta}}^n = M_{\overline{\Theta}}^h, \overline{\pi}_{i\overline{\Theta}}^h = \pi_{i\overline{\Theta}}^h$. QED (5)

Hence:

(6) $\sigma \upharpoonright M_{\overline{\Theta}}^h = \pi_{\overline{\Theta},\Theta}^h$

Proof: Let $x \in M_{\overline{\Theta}}^h, x = \pi_{j\overline{\Theta}}^h(\tilde{x})$ where $j_{T^h} \overline{\Theta}$. Then $\sigma(x) = \sigma(\pi_{j\overline{\Theta}}^h(\tilde{x})) = \pi_{j\Theta}^h(\tilde{x}) = \pi_{\overline{\Theta},\Theta}^h(x)$. QED (6)

(7) Let $h = 0$ or 1. There is $i \geq \overline{\Theta}$ such that $i + 1 <_{T^h} \Theta$ and $i \in A^h$.

Proof: Suppose not. By Lemma **??** we have: $[\overline{\Theta}, \Theta) \cap A^h = \emptyset$. Hence $M_i^n = M_{\overline{\Theta}}^n$ for $\overline{\Theta} \leq i \leq \Theta$. Since $\mathrm{card}(M_{\overline{\Theta}}^h) < \Theta$, ther is then an $i < \Theta$ such that $\nu_i > \mathrm{On} \cap M_i^h$, contradicting the definition of ν_i. QED (7)

Let i_h be the least $i \geq \overline{\Theta}$ such that $i_h + 1 <_{T^h} \Theta$ and $i_h \in A^h (h = 0, 1)$. Assume w.l.o.g. that $i_0 \leq i_1$. Set:

$$J_{\nu_{i_1}}^E =: J_{\nu_{i_1}}^{E^{M_{i_1}^1}} = J_{\nu_{i_1}}^{E^{M_{i_1}^0}}.$$

Then:

$$J_{\nu_{i_0}}^E =: J_{\nu_{i_1}}^{E^{M_{i_0}^0}} = J_{\nu_{i_0}}^{E^{M_{i_0}^1}}.$$

Moreover, if $i_0 < i_1$, then ν_{i_0} is a cardinal in $J^E_{\nu_{i_1}}$. Since $\overline{\Theta} <_{T^n} i_n <_{T^n} \Theta$, we obviously have:

$$\overline{\Theta} = T^n(i_h+1), \overline{\Theta} = \mathrm{crit}(\pi^h_{\overline{\Theta}\Theta}) = \mathrm{crit}(\pi^h_{\Theta,i_h}).$$

Setting: $F^h =: E^{M^h_{i_n}}_{\nu_{i_n}}$, we then have:

(8) $\overline{\Theta} = \mathrm{crit}(F^h)(h=0,1)$

Let $\tau =: \tau_{i_0}$. Then:

(9) $\tau = \tau_{i_n}$.

Proof: $\tau = \overline{\Theta}^{+J^E_{\nu_{i_0}}}$. But $i_0 = i_1$ or ν_{i_0} is a cardinal on $J^E_{\nu_{i_1}}$. Hence $\tau = \overline{\overline{\Theta}}^{+J^E_{\nu_{i_1}}} = \tau_{i_1}$ by acceptability. QED (9)

Since $\overline{\Theta} = T^h(i_h+1)$ we have:

$$J^E_\tau = J^{E^{M_{\overline{\Theta}}}}_\tau \text{ and } \tau \leq \Theta^+ \text{ in } M^h_{\overline{\Theta}}.$$

But then:

(10) $\tau = \overline{\Theta}^{+M^h_{\overline{\Theta}}}$ and $J^E_\tau = H^{M^h_{\overline{\Theta}}}_\tau$.

Proof: If not, i_h+1 would be a truncation point. Hence $\Pi^h_{\overline{\Theta}\Theta}$ would not be a total function on $M^h_{\overline{\Theta}}$, contradicting (6). QED (10)

Hence:

(11) $\mathbb{P}(\overline{\Theta}) \cap M^0_{\overline{\Theta}} = \mathbb{P}(\overline{\Theta}) \cap M^1_{\overline{\Theta}}$.

But then by (6):

(12) $F^h(X) = \Pi^h_{\overline{\theta},i_h+1}(X) = \sigma(X) \cap \lambda^h_{i_n}$ for $X \in \mathbb{P}(\overline{\Theta}) \cap M^h_{\overline{\Theta}}$.

Hence:

(13) $i_0 \neq i_1$.

Proof: Suppose not. Set $i = i_0 = i_1$. By (12) we have:

$$F = F^0 = F^1.$$

Hence:

$$M^0_i || \nu_i = M^1_i || \nu_i = \langle J^E_{\nu_i}, F \rangle,$$

contradicting the definition of ν_i. QED (13)

Hence $i_0 < i_1$ and ν_{i_0} is a cardinal in $J^E_{\nu_{i_1}}$. By (12), however, $F^0 \in J^E_{\lambda_{i_1}}$ by the initial segment condition. But, letting $\pi = \pi^0_{\overline{\Theta},i_{0+1}} \upharpoonright J^E_\tau$, we have: $\langle J^E_{\nu_0}, \pi \rangle$ is the extension of J^E_τ by F^0. Hence $\pi \in J^E_{\lambda_{i_1}}$, since $J^E_{\lambda_{i_1}}$ is a ZFC model. But π maps τ cofinally to ν_{i_0}, where $\lambda_{i_0} > \tau$ is the largest cardinal in $J^E_{\nu_{i_0}}$. Hence ν_{i_0} is not a cardinal in $J^E_{\nu_{i_1}}$.

Contradiction! QED (Lemma 3.5.1)

3.5.3 n–normaliterability

By an *n–normal iteration strategy* we mean a partial function s on normal n–iterations of limit length such that $S(I)$, if defined, is a well founded cofinal branch in $\mathbb{I}$. The concepts *α–successful n–normal strategy* and *n–normally α–iterable* are then defined in the obvious way. M is called *n–normally iterable* iff it is n–normally α–iterable for all α. If M^0, M^1 are premice of cardinals $1 < \Theta$, where Θ is regular, and S^h is a $\Theta+1$–successful n_h–normal iteration strategy for $M^h (h = 0, 1)$, we can define the $\langle n_0, n_1 \rangle$–coiteration of M^0, M^1 *given* by $\langle S^0, S^1 \rangle$ exactly as before. But then the comparison lemma holds for this coiteration by exactly the same proof as before.

3.5.4 Iteration strategy and copying

Lemma 3.5.2. *Let M be normally α–iterable. Let $\sigma : \overline{M} \to_{\Sigma^*} M$. Then $\overline{M}$ is normally α–iterable.*

Proof: Let S be an α–successful strict normal iteration strategy for M. We use the copying procedure and Lemma 3.4.19 to define an α–successful strategy $\overline{S}$ for $\overline{M}$. $\overline{S}$ is defined on the set of strict iterations $\overline{I}$ of $\overline{M}$ having limit length such that σ induces a copy I of $\overline{I}$ onto M with copying maps $\langle \sigma_0 | i < \text{lh}(\overline{I}) \rangle$ which conforms to S. We then set: $\overline{S}(\overline{I}) = S(I)$. $\overline{S}(\overline{I})$ is then a cofinal well founded branch in $\overline{I}$ by Lemma 3.4.19. By induction on $\mu = \text{lh}(\overline{I})$ it then follows that, if $\overline{I}$ is $\overline{S}$–conforming, then σ induces an S–conforming copy I with copying maps $\langle \sigma_i | i < \mu \rangle$. For $\mu = 1$ or limit μ this is trivial. For $\mu = \eta + 1$ where η is a limit, we use the definition of $\overline{S}$. If $\mu = \eta + 1$, we use Lemma 3.4.18 By a virtual repitition of this proof:

Lemma 3.5.3. *Let M be n–normally α–iterable. Let $\sigma : \overline{M} \to_{\Sigma_1^{(n)}} M$. Then $\overline{M}$ is n–normally α–iterable.*

The details are left to the reader.

3.5.5 Full iterability

Normal iterability is too weak a property for many purposes. For instance, we do not kknow, in general, that a normal iterate N of a normally iterable M is itself normally iterable. We therefore introduce the notion of *full iterability*, which is often more useful but, unfortunately, harder to verify.

The process of taking a normal iteration of M can itself be iterated, as can the process of taking a segment of a normal iterate of M. This suggests an expande notion of iteration: Not only normal iterations are allowed, but also (finite or infinite) successions of normal iteration, where the $i+1$ set iteration is applied to a segment of the iterate given by stage i. The formal definition is:

Definition 3.5.11. Let M be a premouse. By a *full iteration I of M of length μ* we mean a sequence $\langle I^i | i < \mu\rangle$ of normal iteration:

$$I^i = si\langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_{h,j}\rangle, T^i\rangle$$

inducing a sequence $M_i = M_i^{M,I}(i < \mu)$ of premice and a commutative sequence of partial maps $\pi_{hj} = \pi_{hj}^{(M,I)}(h \leq j < \mu)$ such that the following hold:

(a) $M_0 = M$.

(b) $M^i_0 \lhd M_i$ for $i < \mu$.

(c) If $i + 1 < \mu$, then I^i has length $l_i + 1$ for some l_i and:

$$M_{i+1} = M^i_{l_i}, \pi_{i,i+1} = \pi^i_{0,l_i}.$$

Call $i < \mu$ a *drop point* in I iff either $M^i_0 \neq M_i$ or $i + 1 < \mu$ and I^i has a truncation on its main branch.

(d) Let $\alpha < \mu$. Then the set of drop points $i < \alpha$ is finite. Moreover, $\pi_{i,\alpha}$ is a total function on M_i whenever $[i, \alpha)$ has no drop point. If α is a limit ordinal then:

$$M_\alpha, \langle\pi_{i\alpha}|i < \mu\rangle$$

is the transitivized direct limit of:

$$\langle M_i|i < \alpha\rangle, \langle\pi_{ij}|i \leq j < \mu\rangle.$$

It is clear that the sequence $\langle M_i|i < \mu\rangle, \langle\pi_{ij}|i \leq j < \mu\rangle$ are uniquely determined by the pair $\langle M, I\rangle$.

Definition 3.5.12. $I = \langle I^i|i < \mu\rangle$ is a full iteration iff it is a full iteration of some M.

Note. We have not excluded the case $\mu = 0$. In this case $I = \emptyset$ is a full iteration of every premouse. We then have: $M^{(N,\emptyset)} = N, \pi^{(N,\emptyset)} = \mathrm{id} \restriction N$.

Definition 3.5.13. Let $I = \langle I^i | i < \mu \rangle$ be a full iteration. The *total length* of I is $\Sigma_{i<\mu} \mathrm{lh}(I^i)$.

Definition 3.5.14. Let I be a full iteration of M. $i < \mu$ is a *truncation point (or drop point)* v with M, I, iff either I^σ is of length $l_i + 1$ and has a truncation on its main branch $T^{i\prime\prime}\{l_i\}$, or else $M_0^i \neq M_i$.

By (d) the set of truncation points $i < \alpha$ is always finite if $\alpha < \mu$ is a limit ordinal.

Definition 3.5.15. I is a full iteration of M *to* M' iff I is a full iteration of M and one of the following holds:

(i) $I = \emptyset$ and $M' = M$

(ii) I has length $\mu = \eta + 1$ and I^η has length $\gamma + 1$, where $M' = M_\gamma^\eta$.

(iii) I has limit length mu, the set of truncation points $i < \mu$ is finite, and:

$$\langle M_i < i < \mu \rangle, \langle \pi_{ij} | i \leq j < \mu \rangle$$

is as the transitive direct limit:

$$M', \langle \pi_i | i < \mu \rangle.$$

Definition 3.5.16. Let M, M', I be as above. The *iteration map* $\pi = \pi^{(M,I)}$ from M to M' given by the pair (M, I) is defined as follows:

(i) $\pi = \mathrm{id} \restriction M$ if $I = \varnothing$

(ii) If I, I^ξ are as in (ii) we set $\pi = \pi_{0,l_\eta}^\eta \circ \pi_{0,\eta}^{(M,I)}$

(iii) If case (iii) holds, we set: $\pi = \pi_0$.

Definition 3.5.17. Let $I = \langle I^i | i < \mu \rangle, I' = \langle I'^i | i < \mu' \rangle$ be full iterations. the *concatenation* $I^\frown I'$ of I, I' is the sequence $\langle \tilde{I}^i | i < \mu + \mu' \rangle$ such that $\tilde{I}^i = I^i$ for $i < \mu$ and $\tilde{I}^{\mu+i} = I'^i$ for $i < \mu'$.

$I^\frown I'$ is not necessarily a full iteration. However, it is easily seen that

Lemma 3.5.4. *If I is a full iteration from M to M' and I' is a full iteration of M', then*

(a) $I^\frown I'$ is a full iteration of M.

(b) If $I' \neq \emptyset$, then $\pi^{(M,I)} = \pi_{0\mu}^{(M,I^\frown I')}$, where $\mu = \mathrm{lh}(I)$.

(c) *If I' is an iteration of M' to M'', then $I^\frown I'$ is an iteration of M to M'' and $\pi^{(M,I^\frown I')} = \pi^{(M',I')} \circ \pi^{(M,I)}$.*

Definition 3.5.18. Let I be a full iteration of M. By a *lenthening* of I we mean any $I^\frown I'$ which is a full iteration.

(Hence we *cannot* lengthen $\langle I^i | i \leq \eta \rangle$ by extending its last normal iteration I^η, but only by starting a new normal iteration.)

Note. Lemma 3.5.4 (b) then says that, if I is an iteration from M to M' and I' is a *proper* lenghtening of I (i.e. $\mu = \text{lh}(I) < \mu' = \text{lh}(I')$, then $\pi^{(M,I)} = \pi_{0\mu}^{(M,I')}$.

We now define the concept of *full iterability*:

Definition 3.5.19. A *full iteration strategy* is a partial function on full iterations I of length $\eta + 1$ such that I^η is of limit length. $S(I)$, if defined is then a cofinal well founded branch in I^η (we refer such full iterations I as *critical*).

Definition 3.5.20. A full iteration $I = \langle I^i | i < \mu \rangle$ *conforms* to the strategy S iff whenever $i < \mu$ and $\gamma < \text{lh}(I^i)$ is a limit ordinal, then $T^{0''}\{\gamma\}$ is the branch $S((I \restriction i)^\frown (I^i | \gamma))$ given by S.

Definition 3.5.21. A strategy S is *α–successful* for M iff whenever $I = \langle I^i | i < \mu \rangle$ is an S–conforming full iteration of M of total length $\Sigma_{i<\mu} \text{lh}(I^i) < \alpha$, then I can be extended one step further in an S–conforming way:

(a) If $\mu = i + 1$ and I^i is of limit length, then $S(I)$ exists.

(b) Let $\mu = i + 1$ and $\text{lh}(I^i) = h + 1$. Extend I^i to a potential normal iteration by appointing ν_h. This gives E_{ν_h} and M_i^*. Then M_h^* is $*$–extendible by E_{ν_h}.

(c) If μ is a limit ordinal, then there are at most finitely many truncation points below μ. Moreover:

$$\langle M_i^{(M,I)} | i < \mu \rangle, \langle \pi_{i,j}^{(M,I)} | i \leq j < \mu \rangle$$

has a well founded limit.

Definition 3.5.22. M is *fully α–iterable* iff it has an α–successful full iteration strategy.

Definition 3.5.23. M is *fully iterable* iff it is fully α–iterable for every α.

3.5.6 The Dodd–Jensen Lemma

We now prove a theorem about normal iteration of premice which are fully iterable and have the normal unique new property.

Theorem 3.5.5. (The Dodd–Jensen Lemma)
Suppose that M has the normal uniqueness property and is fully Θ–iterable, where $\Theta > \omega$ is regular. Let:

$$I^0 = \langle\langle M_i^0\rangle, \langle\nu_i^0\rangle, \langle\pi_{ij}^0\rangle, T^0\rangle$$

be a normal iteration of M with length $\eta + 1$. Let $\sigma : M \to_{\Sigma^*} N$ where $N \lhd M_\eta^0$. Then:

(a) $N = M_\eta^0$.

(b) There is no truncation point on the main branch $T^{0\prime\prime}\{\eta\}$ of I^0.

(c) $\sigma(\xi) \geq \pi_0, (\xi)$ *for all* $\xi \in \mathrm{On} \cap M$.

Note. Let $M' = M_\eta^0, \pi = \pi_{0,\eta}$. Then π is the unique Σ^*–preserving map of M to M' such that $\pi(\xi) =$ the least ξ' such that $\xi' = \sigma(\xi)$ for some $\sigma : M \to M'$ which is Σ^*–preserving. Thus π depends only on the models M, M' and not on the iteration I^0.

We now prove the theorem. Fix a Θ–successful strategy S for M. By induction on $i < \omega$ we construct I^i, N^i, σ^i such that

- $I^i = \langle\langle M_h^i\rangle, \langle\nu_h^i\rangle, \langle\pi_{hj}^i, T^i\rangle$ is a normal iteration.
- $N^i \lhd M_\eta^i$ and $\sigma^i : M \to_{\Sigma^*} N^i$.
- $\langle I^0, \ldots, I^i\rangle$ is S–conforming.
- If $i = h + 1$, then I^i is the copy of I^0 onto N^h by σ^h.

Case 1 $i = 0$
I^0 is given. Set: $N^0 = N, \sigma^0 = \sigma$.

Case 2 $i = h + 1$
We first construct I^i. We construct $I^i|\gamma + 1$ and copying maps

$$\sigma_l^h : M_l^0 \to_{\Sigma^*} M_l^0 (l \leq \gamma)$$

by induction on γ, ensuring at each stage that $\langle I^0, \ldots, I^h, I^i|\gamma + 1\rangle$ is S–conforming.

For $\gamma = 0$ set $I^i|\gamma+1 = \langle\langle N^h\rangle, \varnothing, \langle\,\mathrm{id}\,\rangle, \varnothing\rangle$. We set $\sigma_0^h = \sigma^h$. If $\gamma = l+1$, we follow the usual procedure.

Now let γ be a limit ordinal. We are given $I^i|\gamma$ and copying maps $\langle\sigma_l^h | l < \gamma\rangle$, where $I^i|\gamma$ is the copy of $I^0|\gamma$ onto $M_0^i = N^h$ by σ^h. Then $I' = \langle I^0, \ldots, I^h, I^i|\gamma\rangle$ is S–conforming. Hence S gives us a cofinal well founded branch $b = S(I')$ in $I^i|\gamma$ and we extend $I^i|\gamma$ to $I^i|\gamma+1$ by setting $T^{i\prime\prime}\{\gamma\} = B$. But by Lemma 3.4.19, b is a well founded cofinal branch in $I^0|\gamma$. Hence $b = T^{0\prime\prime}\{\gamma\}$ by uniqueness. But then $\sigma_{\gamma+1}^i : M_\gamma^0 \to M_\gamma^i$ can be defined as usual. This gives $\langle I^0, \ldots, I^i\rangle$, which is S–conforming. But $\sigma_\eta^h : M_\eta^0 \to_{\Sigma^*} M_\eta^i$, where $N^0 \lhd M_\eta^0$. If $N^0 = M_\eta^0$, set $N^i = M_\eta^i$. Otherwise set: $N^i = \sigma_\eta^h(N^0)$. In either case $\sigma_\eta^h \cdot \sigma^0 : M \to_{\Sigma^*} N^i$, and we set: $\sigma^i = \sigma_\eta^h \cdot \sigma^0$. QED (Case 2)

Thus $\langle I^i | i < \omega\rangle$ is an S–conforming full iteration of M. Using this we prove (a) – (c):

(a) Suppose not. Then $N^i \neq M^i$ for $i < \omega$. But $M_0 = M, M_{n+1} = M_\eta^n$ and $M_0^{n+1} = N^n \neq M_{n+1}$. Hence every $n+1 < \omega$ is a truncation point in $I = \langle I^n | n < \omega\rangle$.
Contradiction!

(b) Suppose not. Let $i+1$ be a truncation point on the main branch $T^{0\prime\prime}\{\eta\}$ of I^0. By our construction $i+1$ is a truncation point in $T^{n\prime\prime}\{\eta\}$ for $n < \omega$. Hence each $n+1$ is a truncation point in I.
Contradiction!

(c) By (a), (b), $\pi_{nm} : M_n \to M_m$ is a total function on M_n for $n \leq m < \omega$. Suppose (c) to be false. Let $\sigma^0(\xi) < \pi_0^0(\xi)$. Then $\sigma^{i+1}(\xi) = \sigma_\eta^i(\sigma^0(\xi) < \sigma_\eta^i(\pi_{0\eta}^0(\xi)) = \pi_{0\eta}^i(\sigma^i(\xi)) = \pi_{i,i+1}^{(M,I)}(\sigma^i(\xi))$. Hence $\pi_{i+1,\omega}\sigma^{i+1}(\xi) < \pi_{i,\omega}\sigma^i(\xi)$ for $i < \omega$.
Contradiction! QED (Theorem 3.5.5)

Lemma 3.5.6. *Let $\omega < \Theta \leq \alpha$ where Θ is a regular cardinal. Let S be an α–successful strategy for M. Let I be an S–conforming iteration from M to M' with total length $< \Theta$. Define an iteration strategy S' for M' by*

$$S'(I') \simeq S(I^\frown I')$$

for full iteration I' of M'. Then S' is an α–successful strategy for M'.

The proof is left to the reader. Similarly, we obtain a normal iteration strategy S'' for M by setting S'' for M by setting $S''(I) \simeq S'(\langle I\rangle)$ where I is a normal iteration of limit length $< \alpha$ and $\langle I\rangle$ is the full iteration $\tilde{I}$ of length 1 such that $\tilde{I}^0 = I$.

3.5.7 Copying a full iteration

Definition 3.5.24. Let $\sigma : M \rightarrow_{\Sigma^*} M'$ where M, M' are premice. Let $I = \langle I^i | i < \mu \rangle$ be a full iteration of M. $I' = \langle I'^i | i < \mu \rangle$ is the *copy* of I *onto* M' by σ with copying maps $\langle \sigma^i < i < \mu \rangle$ iff

(a) I' is a full iteration of M' inducing

$$\langle M'_i | i < \mu \rangle, \langle \pi'_{ij} | i \leq j < \mu \rangle$$

(b) $\sigma_i : M_i \rightarrow_{\Sigma^*} M'_i$ such that $\sigma_i \pi_{ij} = \pi'_{ij} \sigma_i$

(c) $\sigma_0 = \sigma$

(d) I'^i is the copy of I^i induced by $\sigma_i \upharpoonright M_0^i$ with copying maps $\langle \sigma_h^i | h < \text{lh}(I^i) \rangle$

(e) If $M_i = M_0^i$, then $M'_i = M'^i_0$ and $\sigma^i = \sigma_0^i$.

(f) If $M_i \neq M_0^i$, then $M'^i_0 = \sigma_i(M_0^i)$ and $\sigma_0^i = \sigma_i \upharpoonright M_0^i$

(g) If $i + 1 < \mu$, then $\sigma_{i+1} = \sigma_{li}^i$ where $\text{lh}(I^i) = l_i$.

Clearly I' and the copying maps $\langle \sigma_i | i < \mu \rangle, \langle \sigma_h^i | i < \mu, h < \text{lh}(I^i) \rangle$ are unique, if they exist. (Note that if $\eta < \mu$ is a limit ordinal, then σ_η is uniquely defined by: $\sigma_\eta \pi_{i\eta} = \pi'_{i\eta} \sigma_i$ for $i < \eta$.)

Lemma 3.5.7. *Let $\sigma : M \rightarrow_{\Sigma^*} M'$, where M' is fully α–iterable. Then M is fully α–iterable.*

Let S' be an α–successful strategy for M'. We define a strategy S for M as follows: If $I = \langle I^i | i \leq \eta \rangle$ is a full iteration of M such that I^η is of limit length, we ask whether σ induces a copy I' of I onto M'. If so we set: $S(I) \simeq S'(I')$. If not, $S(I)$ is undefined. ($S(I)$, if defined, is a cofinal well founded branch in I^η by Lemma 3.4.19.) It follows that if I is S–conforming, then σ induces a copy I' which is S'–conforming. (We prove this by induction on μ, where $I = \langle I^i | i < \mu \rangle$ and for $\mu = \eta + 1$ by induction on the length of I^η.) Using Lemma 3.4.18 and 3.4.19 it then follows that I can be extended in an S–conforming way, since I' can be extended in an S'–conforming way.
QED (Lemma 3.5.7)

3.5.8 The Neeman–Steel lemma

The usefulness of the Dodd–Jensen Lemma is limited by the fact that it applies only to premice with the normal uniqueness property. In the absence of normal uniqueness we have the following subtleties:

Theorem 3.5.8 (The Neeman–Steel Lemma). *Let M be a countable premouse which is fully $\omega+1$ iterable. Let $\langle \xi_n | n < \omega\rangle$ be an enumeration of* $\mathrm{On} \cap M$*. There is an ω_1-successful full iteration strategy S for M such that whenever $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ is an S-conforming normal iteration of M of length $\eta+1 < \omega_1$ and $\sigma : M \longrightarrow_{\Sigma^*} M'$, where $M' \lhd M_\eta$, then:*

(a) *$M' = M_\eta$.*

(b) *There is no truncation point on the main branch $\{i : iT\eta\}$.*

(c) *If $\sigma(\xi_i) = \pi_{0,\eta}(\xi_i)$ for $i \leq n < \omega$, then $\sigma(\xi_n) \geq \pi_{0,\eta}(\xi_n)$.*

Then $\pi_{0,\eta}$ is the unique $\pi : M \longrightarrow_{\Sigma^*} M'$ such that $\pi(\xi_n) =$ the least ξ' such that $\sigma(\xi_n) = \xi'$ for a σ such that $\sigma : M \longrightarrow_{\Sigma^*} M'$ and $\sigma(\xi_i) = \pi(\xi_i)$ for $i < n$. Then π depends only on M, M' and the enumeration $\langle \xi_i : i < \omega\rangle$, rather than on the iteration I.

Note. When we say that a normal iteration is S-conforming, we mean that the full iteration $\langle I\rangle$ of length 1 is S-conforming.

We shall derive Theorem 3.5.8 from a stronger statement:

Lemma 3.5.9. *Let $M, \langle \xi_i : i < \omega\rangle$ be as above. There is a ω_1+1-successful full iteration strategy S for M such that whenever I is an S-conforming full iteration from M to M' and $\sigma : M \longrightarrow_{\Sigma^*} M'$, then:*

(a) *No $i < \mathrm{lh}(I)$ is a drop point in I (hence the iteration map π from M to M' is a total function on M).*

(b) *If $\sigma(\xi_i) = \pi(\xi)$ for $i < n$, then $\sigma(\xi_n) \geq \pi(\xi_n)$.*

This clearly implies Theorem 3.5.8 since if $I = \langle\langle M_i\rangle, m\rangle$, M' are as in the theorem, then $\langle I, \langle M'\rangle\rangle$ is an S-conforming full iteration from M to M' of length 2. (Here $\langle M\rangle$ denotes the minimal normal iteration of M of length 1: $\langle M\rangle, \varnothing, \langle \mathrm{id} \restriction M\rangle, \varnothing\rangle$.)

Proof. We prove Lemma 3.5.9. In the following we use the term "iteration" to mean a full iteration of total length $< \omega_1$. By a *lengthening* of an iteration

I we mean an iteration of the form $I^\frown I'$. Fix an ω_1+1-successful iteration strategy for M. We write "S-iteration" to mean "S-conforming iteration".

(1) There is an iteration I_0 from M to an N_0 such that:

- There is $\sigma_0 : M \longrightarrow_{\Sigma^*} N_0$.
- Let I be any lengthening of I_0 which is an S-iteration from M to M'. Let $\sigma' : M \longrightarrow_{\Sigma^*} M'$. Then I has no truncation point in $\mathrm{lh}(I) \setminus \mathrm{lh}(\hat{I})$.

Proof. Suppose not. Recall that $\varnothing$ is an S-iteration of M to M. There is then a sequence of $\langle I_i, N_i, \sigma_i\rangle (\sigma < \omega)$ such that:

- $I_0 = \varnothing, N_0 = M, \sigma_0 = \mathrm{id}\, M$.
- $I_i + 1$ is an S-iteration of M to $N_i + 1$ which lengthen Ii.
- $I_i + 1$ has a truncation point in $\mathrm{lh}(I_i + 1) \setminus \mathrm{lh}(I_i)$.
- $\sigma_i : M \longrightarrow_{\Sigma^*} N_i$.

Set $I = \bigcup_i I_i$. Then I is an S-iteration with infinitely many truncation points below $\mathrm{lh}(I)$. Contradiction!

QED (1)

Fix I_0, N_0, σ_0.

(2) We can extend $\langle I_0, N_0, \sigma_0\rangle$ to an infinite sequence $\langle I_i, N_i, \sigma_i\rangle$ $(i < \omega)$ such that:

- $I_i = I_h^\frown I_{h,i}$ is an S-iteration which lengthen I_h for $h < i$.
- $I_{h,i}$ is an iteration from N_h to N_i with iteration map $\pi_{h,i} = \pi^{(N_h, I_{h,i})}$.
- $\pi_{ij}\pi_{hi} = \pi_{hi}$ for $h \leq i \leq j < \omega$.
- $\sigma_i : M \longrightarrow_{\Sigma^*} N_i$
- $\pi_{ij}\sigma_i(\xi_h) = \xi_h$ for $h < i < j$.
- Let $j = i+1$ and let $I_j^\frown I$ be any S-iteration, where I is from N_j to N. Let $\sigma : M \longrightarrow_{\Sigma^*} N$ such that $\sigma(\xi_h) = \pi\sigma_j(\xi_h)$ for $h < j$, where $\pi = \pi^{(N_j, I)}$ is the iteration map. Then $\sigma(\xi_i) \geq \pi\sigma_j(\xi_i)$.

Proof. Suppose not. Consider the tree of finite sequences $\langle\langle I_i, N_i, \sigma_0\rangle : i \leq n\rangle$ such that the above holds for all $h, i, j \leq n$. This tree has no infinite branch. Hence there is a finite sequence $\langle\langle I_i, N_i, \sigma_i\rangle : i \leq n\rangle$ which has no successor in the tree. Nut then we can form a sequence

$$\langle \tilde{I}_i, \tilde{N}_i, \tilde{\sigma}_i\rangle,\ i \leq \omega$$

with the properties:

- $\tilde{I}_0 = I_n, \tilde{N}_0 = N_n, \tilde{\sigma}_0 = \xi_n$.
- $\tilde{I}_{i+1} = \tilde{N}_i^\frown \tilde{I}_i'$ is an S-iteration from M to $\tilde{N}_{i+1}$ which properly lengthens $\tilde{N}_i$.
- $\tilde{I}_i'$ is an iteration from $\tilde{N}_i$ to $\tilde{N}_{i+1}$ with iteration map $\pi_i = \pi^{(\tilde{N},\tilde{I}_i')}$.
- $\tilde{\xi}_{i+1} : M \longrightarrow_{\Sigma^*} \tilde{N}_{i+1}$ is such that $\tilde{\xi}_{i+1}(\xi_h) = \pi\tilde{\xi}_i(\xi_h) = \pi_i\tilde{\xi}_i(\xi_h)$ for $h < n$ but $\tilde{\xi}_{i+1}(\xi_n) < \pi_i(\tilde{\xi}_i(\xi_n))$.

Set $\mu_i = \mathrm{lh}(\tilde{I}_i)$, $\tilde{I} = \bigcup_i I_i$. Then $\mu_i < \mu_{i+1}$ and $\tilde{I}$ is of limit length $\mu = \sup_i \mu_i$ since $\tilde{I}_i$ lengthens I_0 and $\tilde{\sigma}_i : M \longrightarrow_{\Sigma^*} \tilde{N}_i$. Let $M_l = M_l^{(M,\tilde{I})}, \tilde{\pi}_{l,j} = \pi_{l,j}^{(M,\tilde{I})}$ for $l \leq i < \mu$, it follows easily that $\pi_i = \tilde{\pi}_{\mu_i,\mu_{i+1}}$ and $\tilde{N}_i = M_i$. Moreover $\tilde{\pi}_{\mu_i,j}$ is a total function on M_i for $\mu_i \leq j < \mu$. Since $\tilde{I}$ is S-conforming we can form the transitive limit $\tilde{M}, \langle \tilde{\pi}_i : i < \mu\rangle$ of:

$$\langle M_i : i < \mu\rangle, \langle \pi_{i,j} : i \leq j < \mu\rangle.$$

But then $\tilde{\pi}_{\mu_i+1}\tilde{\sigma}_{i+1}(\xi_n) < \tilde{\pi}_i\tilde{\sigma}_i(\xi_n), i < \omega$. Contradiction!

QED(2)

Now let $\langle I_i, N_i, \sigma_i\rangle, i < \omega$ be as in (2). Let $\mu_i =: \mathrm{lh}(I_i)$. We assume without lose of generality that $\mu_i < \mu_j$ for $i < j$. If I_i' is an S-iteration from M to M', then so if $I'^\frown\langle M'\rangle$. Set $I^* = \bigcup_i I_i$. I^* is an S-iteration of length $\mu^* = \sup_i \mu_i$. We know by (1) that I^* has no truncation point in $\mu^* \setminus \mu_0$. Letting $M^* = M_i^{M,I^*}, \pi_{i,j}^* = \pi_{i,j}^{(M,I^*)}$, we have:

$$N_i = M_{\mu_i}^* \text{ and } \pi_{ij} = \pi_{\mu_i,\mu_j}^*$$

where N_i, π_{ij} are as in (2). Since I^* is an S-iteration, we can form the limit:

$$M^*, \langle \pi_i^* : i < \mu^*\rangle$$

of $\langle M_i^* : i < \mu^*\rangle, \langle \pi_{ij}^* : i \leq j < \mu^*\rangle$. But $\pi_{\mu_{i+1}}^*(\sigma_{i+1}(\xi_h)) = \pi_{\mu_{j+1}}^*(\sigma_{j+1}(\xi_h))$ for $h \leq i \leq j < \omega$, where $\sigma_{i+1} : M \to M$ and $\pi_{\mu_{i+1}}^* : M_{\mu_{i+1}} \to M^*$ are Σ^*–preserving. But then we can define a $\sigma^* : M \to_{\Sigma^*} M^*$ by:

$$\sigma^*(\xi_n) = \pi_{\mu_{i+1}}(\sigma_{i+1}(\xi_h)) \text{ for } h \leq i < \omega.$$

Let S^* be the $\omega_1 + 1$–successful strategy for M^* defined by:

$$S^*(I) \simeq S(I^{*\frown} I)$$

where I is any full iteration of M^*. Following the prescription in the proof of Lemma **??** we can then define a strategy $\overline{S}$ for M by: If $\overline{I}$ is an iteration of M, we first ask wheter σ^* induces a copy I of $\overline{I}$ onto M^*. If so we set:

$$\overline{S}(\overline{I}) \simeq S^*(I) \simeq S(I^{*\frown} I).$$

If $\overline{I}$ is $\overline{S}$–conforming, it follows that I is S^*–conforming, hence that $I^* \frown I$ is S–conforming. Using this, we show that $\overline{S}$ satisfies (a), (b). Let $\overline{I}$ be an iteration from M to $\overline{M}$ and let $\overline{\sigma} : M \to_{\Sigma^*} \overline{M}$. σ^* induces an iteration I from M^* to M' with copying map $\sigma' : \overline{M} \to M'$. Thus $\sigma'\overline{\sigma} : M \to_{\Sigma^*} M'$. Let $\overline{\pi} = \pi^{(M,\overline{I})}$ be the iteration map from M to $\overline{M}'$. Let $\pi = \pi^{(M^*,I)}$ be the iteration map from M^* to M'. Then $\sigma'\overline{\pi} = \pi\sigma^*$, since σ' is a copying map.

(3) There is no truncation point $i < \text{lh}(\overline{T})$.

Proof. Suppose not. Then i is a truncation point in I and $\mu^* + i$ is a truncation point in $I^* \frown I$, contradicting (1), since $\sigma'\overline{\sigma} : M \to_{\Sigma^*} M'$.

QED (3)

(4) Let $\overline{\sigma}(\xi_h) = \overline{\pi}(\xi_h)$ for $h < i$. Then $\overline{\sigma}(\xi_i) \geq \overline{\pi}(\xi_i)$.

Proof. Suppose not. Note that

$$\sigma'\overline{\pi}(\xi_h) = \pi\sigma^*(\xi_h) = \pi\pi_{\mu^*_{i+1}}\sigma_{i+1}(\xi_h)$$

for $h \leq i$. But $I^* \frown I = I_{i+1} \frown \tilde{I}$ where $\tilde{I}$ is an iteration from N_{i+1} to N with iteration map $\tilde{\pi} = \pi^{(N_{i+1},\tilde{I})}$. It is easily seen that $\tilde{\pi} = \pi\pi_{\mu^*_{i+1}}$, hence

$$\sigma'\overline{\pi}(\xi_h) = \tilde{\pi}\sigma_{i+1}(\xi_h) \text{ for } h \leq i.$$

Hence $\sigma'\overline{\sigma}(\xi_h) = \tilde{\pi}\sigma_{i+1}(\xi_h)$ for $h < i$, but

$$\sigma'\overline{\sigma}(\xi_i) < \sigma'\overline{\pi}(\xi_i) = \tilde{\pi}\sigma_{i+1}(\xi_i).$$

This contradicts (2).

QED(4)

This proves Lemma 3.5.9 and with it Theorem 3.5.8.

QED(Lemma 3.5.9)

QED(Theorem 3.5.8)

The fact that the Neeman-Steel lemma holds only for countable mice is a less serious limitation than one might suppose. In practice, both the Dodd–Jensen lemma and the Newman–Steel lemma are used primarily to establish properties of mice which - by a Löwenheim-Skolem argument - hold generally if they hold for countable mice.

3.5.9 Smooth iterability

Definition 3.5.25. By a *smooth iteration of M* we mean a full iteration I of M such that $M_i = M_0^i$ for $i < \mathrm{lh}(I)$.

The concepts "smooth iteration strategy", "i–successful smooth iteration strategy" and "smooth α–iterable" are defined accordingly. We shall eventually prove that every smoothly iterable premouse is fully iterable. The proof will depend on enhanced copying procedures.

3.5.10 n–full iterability

We said at the outset that a "mouse" will be defined to be a premouse which is iterable. But what is the right notion of iterability? full iterability feels right. An, indeed, we shall ultimately show that, if there is no inner model with a Woodin cardinal, then every normally iterable premouse is fully iterable. However, it will take a long time to reach that point, and in the meantime we must make do with weaker forms of iterability which are easier to verify. The main problem will be this. Our procedure for verifying that a premouse M is normally iterable will not show that normal iterates of M are themselves iterable. What it will show is weaker: If, by an appropriate strategy, I is a normal iteration of M to M' of length $\eta + \delta$ *and* if $\rho^n_{M'} > \lambda_i$ for $i < \eta$, then M' ia n–normally iterable. For this reason we will often be forced to work with n–iteration rather than $*$–iterations, and we must employ a sharply restricted notion of "full iteration". We define:

Definition 3.5.26. Let I be an m–normal iteration of length $\eta+1$ for some $m \leq \omega$. Let $n \leq \omega$. I is *n–bounded* iff $\lambda_i < \rho^n_{M_2}$ for all $i < \eta$.

Definition 3.5.27. I is an *m to n–normal iteration* iff I is an n–bounded m–normal iteration.

We shall be mainly interested in n to n iterations.

Definition 3.5.28. Let M be a premouse. Let $n \leq \omega$ by an *n–full iteration i of length μ* we mean a sequence $\langle I^i | i < \mu\rangle$ of n–normal iterations such that I^i is n to n normal for $i + 1 < \mu$, inducing a sequence $M_i = M_i^{(M,I)} (i < \mu)$ of premice and a commutative sequence $\pi_{ij} = \pi_{ij}^{(M,I)}$ of partial maps from M_i to $M_j (i \leq j < \mu)$ satisfying (a) – (d) of our previous definition.

Note. If $I = \langle I^i | i \leq \eta\rangle$ is an n–full iteration of length $\eta + 1$, then the final n–normal iteration I^η is not neccessarily n to n, though the previous ones are. However, if I^η is not n to n, then there is no possibility of lengthening the sequence I, thouch I^η itself could be lengthened.

We can take over our previous definitions — in particular the definition of "n–full iteration from M to N" and "n–full iteration map" $\pi^{M,I}$.

Definition 3.5.29. $I = \langle I^i | i < \eta \rangle$ is an *n to n full* iteration if I is n–full and each I^i is an n to n–normal iteration.

The definition of "concatenation" is as before. It is cler that if I is an n to n–full iteration from M to M' and I' is an n–full iteration of M', then $I^\frown I'$ is an n–full iteration of M.

Lemma 3.5.4 holds as before, on the assumption that I is an n to n–full iteration from M to M' and I is an n–full iteration of M. THe concepts *n–full iteration strategy* is defined as before, as is the concept of an *S–conforming* n–full iteration, *α–successful* n–full strategy, and n–full *α–iterability.*

The Dodd–Jensen lemma then holds in the form:

Theorem 3.5.10. *Suppose that M has the n–normal uniqueness property and is n–fully Θ–iterable, where $\Theta > \omega$ is regular. Let:*

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be an n to n–normal iteration of M with length $\eta + 1$. Let $\sigma : M \to_{\Sigma^*} N$ where $N \triangleleft M_\eta$. Then:

(a) $N = M_\eta$.

(b) There is no truncation point on the main branch $T''\{\eta\}$ of I.

(c) $\sigma(\xi) \geq \pi_{o,\eta}(\xi)$ *for all* $\xi \in \mathrm{On} \cap M$.

The proof is a virtual repetition of the previous proof.

Lemma 3.5.6 holds *mutatis mutandis* just as before. We define what it means for $\sigma : M \to_{\Sigma^{(n)}} M'$ to *induce a copy I' of I onto M' with copying maps $\langle \sigma^i \rangle$* just as before, writing $\Sigma^{(n)}$ instead of Σ^* everywhere.

Theorem 3.5.11. *Let M be a countable premouse which is n–fully $\omega_1 + 1$ iterable. Let $\langle \xi_n | n < \omega \rangle$ be an enumeration of $\mathrm{On} \cap M$. There is an $\omega_1 + 1$–successful n–full iteration strategy S for M such that whenever $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, \tau \rangle$ is an S–conforming n to n–normal iteration of M of length $\eta + 1 < \omega_1$ and $\sigma : M \to_{\Sigma^{(n)}} M'$ where $M' \triangleleft M_\eta$, then:*

(a) $M' = M_\eta$.

(b) There is no truncation point on the main branch $\{i | iT_\eta\}$.

(c) *If* $\sigma(\xi_i) = \pi_{0,\eta}(\xi_i)$ *for* $i < n < \omega$*, then* $\sigma(\xi_n) \geq \pi_{0,\eta}(\xi_n)$.

As before, this follows from:

Lemma 3.5.12. *Let* $M, \langle \xi_i | i < \omega \rangle$ *be as above. There is an* $\omega_1 + 1$*–successful* n*–full iteration strategy* S *to* M *such that whenever* I *is an* S*–conforming* n *to* n*–full iteration from* M *to* M' *and* $\sigma : M \rightarrow_{\Sigma^{(n)}} M'$*, then:*

(a) *No* $i < \mathrm{lh}(I)$ *is a truncation point. (Hence the map* $\pi = \pi^{(M,I)}$ *is a total function on* M*.)*

(b) *If* $\sigma(\xi_i) = \pi(\xi_i)$ *for* $i < n$*, then* $\sigma(\xi_n) \geq \pi(\xi_n)$.

The proofs are virtually unchanged.

3.6 Verifying full iterability

3.6.1 Introduction

As we said, full iterability is a difficult property to verify. A theorem that every normally iterable mouse is fully iterable would be useful, if true, but seems unlikely. We can, however, prove the following pair of theorems:

Theorem 3.6.1. *If* M *is smoothly* α*–iterable, then it is fully* α*–iterable.*

Theorem 3.6.2. *Let* $\kappa > \omega$ *be regular and let* M *be uniquely normally* $\kappa+1$ *iterable. Then* M *is smoothly* $\kappa + 1$*–iterable.*

The proofs of these theorems are quite complex. To prove theorem 3.6.1, we redo much of chapter 2, developing a theory of embeddings which are Σ^*–preserving *modulo pseudo projecta*, which may not be the real projecta, but behave simiarly. The proof of theorem 3.6.2 requires us, in addition, to delve rather deeply into the combinatorics of normal iteration, using technique which, essentially, were developed by John Steel and Farmer Schlutzenberg.

This section (§3.6) is devoted to the proof of theorem 3.6.1. The following section brings the proof of theorem 3.6.2. In later chapters we shall make frequent use of both these theorems, but will seldom, if ever, refer to their proofs. Hence it would be justifiable for a first time reader of this this book to skip §3.6 and §3.7, taking the above theorems for granted and deferring their proofs until later.