is the S-iteration map from N to $\hat{N}$. Hence $\sigma'\pi^1(e_i) < \sigma'\pi^0(e_i)$, since $\sigma'\pi^0 : N \longrightarrow_{\Sigma^*} \hat{N}$. Hence $\pi^1(e_i) < \pi^0(e_i)$. Contradiction!

QED(Claim)

Let $i_h + 1 \leq_{T^h} \mu$ with $o = T^h(i_h+1)$ for $h = 0, 1$. Then $\kappa_{i_0} = \kappa_{i_1} = \text{crit}(\pi)$, where $\pi = \pi^0_{0,\mu} = \pi^1_{0,\mu}$. Set:

$$F^0 = E^{Q_0}_{\nu_{i_0}}, F^1 = E^{M_0}_{\nu_{i_1}}.$$

Then:

$$F^h(X) = \pi^h_{0,i_h+1}(X) \text{ for } X \in \mathbb{P}(\kappa_{i_h}) \cap N.$$

Thus:

$$\alpha \in F^h(X) \longleftrightarrow \alpha \in \pi(X) \text{ for } \alpha < \lambda_{i_h},$$

since $\pi = \pi^h_{i_h+1,\mu} \circ \pi^h_{0,i_h+1}$. But then $\nu_{i_0} \not< \nu_{i1}$, since otherwise $F^0 \in J^{E^{M_{i_1}}}_{\nu_{i_1}}$ by the initial segment condition, whereas ν_{i_0} is a cardinal in $J^{E^{M_{i_1}}}_{\nu_{i_1}}$. Contradiction! Similarly $\nu_{i_1} \not< \nu_{i_0}$. Thus $i_0 = i_1 = i$ and $F^0 = F^1$. But then ν_i is not a coiteration index! Contradiction.

QED(Claim 4)

This proves the simplicity lemma.

4.3 Solidity and Condensation

In this section we employ the simplicity lemma to establish some deep structural properties of mice. In §4.3.1 we prove the **Solidity Lemma** which says that every mouse is solid. In §4.3.2 we expand upon this showing that any mouse N has a unique core $\overline{N}$ and core map σ defined by the properties:

- $\overline{N}$ is sound.
- $\sigma :\longrightarrow_{\Sigma^*} N$.
- $\rho^\omega_{\overline{N}} = \rho^\omega_N$ and $\sigma \upharpoonright \rho^\omega_N := \text{id}$.
- $\sigma(p^i_{\overline{N}}) = p^i_N$ for all i.

In §4.3.3 we consider the *c*ondensation properties of mice. The condensation lemma for L says that if $\pi : M \longrightarrow_{\Sigma_1} J_\alpha$ and M is transitive, then $M \triangleleft J_\alpha$. Could the same hold for an arbitrary sound mouse in place of J_α? In that generality it certainly does not hold, but we discover some interesting instances of condensation which do hold.

We continue to restrict ourselves to premice M such that $M||\alpha$ is not of type 3 for any α. By a mouse we mean such a premouse which is fully iterable. (Though we can take this as being relativized to a regular cardinal $\kappa > \omega$, i.e. $\mathrm{card}(M) < \kappa$ and M is fully $\kappa + 1$-iterable.)

4.3.1 Solidity

The *Solidity lemma* says that every mouse is solid. We prove it in the slightly stronger form:

Theorem 4.3.1. *Let N be a fully $\omega_1 + 1$-iterable premouse. Then N is solid.*

We first note that we may w.l.o.g. assume N to be countable. Suppose not. Then there is a fully $\omega_1 + 1$ iterable N which is unsolid, even though all countable premice with this property are solid. Let $N \in H_\theta$, where θ is a regular cardinal. Let $\sigma : \overline{H} \prec H_\theta$, $\sigma(\overline{N}) = N$, where $\overline{H}$ is transitive and countable. Then $\overline{H}$ is a ZFC^- model. Since $\sigma \restriction \overline{N} : \overline{N} \prec N$, it follows by a copying argument that $\overline{N}$ is a $\omega_1 + 1$ fully iterable (cf. Lemma 3.5.6.). Hence $\overline{N}$ is solid. By absoluteness, $\overline{N}$ is solid in the sense of $\overline{H}$. Hence N is solid in the sense of H_θ. Hence N is solid. Contradiction!

Now let $a = P^n_N$ for some $n < \omega$. Let $\lambda \in a$. Let $M = N^\lambda_a$ be the λ−th witness to a as defined in §4.1. For the reader's convenience we repeat that definition here. Let:

$$\rho^{l+1} \leq \lambda < \rho^l \text{ in } N;\ b =: a \setminus (\lambda + 1)$$

Let $\overline{N} = N^{l,b}$ be the l-th reduct of N by b. Set:

$$X = h(\lambda \cup b) \text{ where } h = h_{\overline{N}} \text{ is the } \Sigma_1\text{-Skolem function of } \overline{N}.$$

Then $X = h``(\omega \times (\lambda \times \{b\}))$ is the smallest Σ_1-closed submodel of $\overline{N}$ containing $\lambda \cup b$. Let:

$$\overline{\sigma} : \overline{M} \longleftrightarrow \overline{N}|X \text{ where } \overline{M} \text{ is transitive.}$$

By the extension of embedding lemma, there are unique $M, \sigma, \overline{b}$ such that $\sigma \supset \overline{\sigma}$ and:

$$\overline{M} = M^{l,b},\ \sigma : M \longrightarrow_{\Sigma_1'} N \text{ and } \sigma(\overline{b}) = b.$$

Then $N^\lambda_a =: M$ and $\sigma^\lambda_a =: \sigma$.

It is easily seen that σ witnesses the phalanx $\langle N, M, \lambda\rangle$. Employing the simplicity lemma, we coiterate $\langle N, M, \lambda\rangle$ against N, getting $\langle I^N, I^M\rangle$, terminating at η, where:

- $I^N = \langle\langle N_i\rangle, \langle\nu_i^N\rangle, \langle\pi_{ij}^N\rangle, T^N\rangle$ is the iteration of N.
- $I^M = \langle\langle M_i\rangle, \langle\nu_i^M\rangle, \langle\pi_{ij}^M\rangle, T^N\rangle$ is the iteration of $\langle N, M\rangle$.
- $\langle\nu_i : i < \eta\rangle$ is the sequence of coiteration indices. We know that:
- $M\eta \triangleleft N_\eta$.
- I^M has no truncation on its main branch.
- $1 \leq_{T^M} \eta$.

It follows that $\kappa_i \geq \lambda$ for $i <_{T^M} \eta$. Moreover $\nu_i > \lambda$ for $i < \eta$, since $M|\lambda = N|\lambda$.

We consider three cases:

Case 1. $M_\eta = N_\eta$ and I^N has no truncation on its main branch.

We know that $\rho_M^{l+1} \leq \lambda$, since every $x \in M$ is $\Sigma_1^{(l)}(M)$ in $\lambda \cup \bar{b}$. But $\kappa_i \geq \lambda$ for $i <_{T^M} \eta$.

Hence:

(1) $\mathbb{P}(\lambda) \cap M = \mathbb{P}(\lambda) \cap M_\eta$ and $\rho_M^h = \rho_{M_\eta}^h$ for $h > i$. But then $\kappa_j \geq \rho_N^{l+1}$ for $j <_{T^N} \eta$, since otherwise:

$$\kappa_i < \sup \pi_{h,j+1}^N{}''\rho_N^{l+1} \leq \rho_{N_\eta}^{l+1} = \rho_{M_\eta}^{l+1} \leq \lambda < \kappa_j$$

where $h = T^N(j+1)$. Hence for $h > l$ we have:

(2) $\rho_M^h = \rho_N^h$ and $\mathbb{P}(\rho^h) \cap M = \mathbb{P}(\rho^h) \cap N$.

Recall, however, that $a = p_N^n$, where $m > l$. Since every $x \in M$ is $\Sigma_1^{(i)}(M)$ in $\lambda \cup \bar{b}$, there is a finite $c \subset \lambda$ such that $c \cup \bar{b} \in P_M^n$. Let $\overline{A}$ be $\Sigma_1^{(n)}(M)$ in $c \cup \bar{b}$ such that $\overline{A} \cap \rho^n \notin M$. Let A be $\Sigma_1^{(n)}(N)$ in $c \cup b$ by the same definition. Then:

$$\overline{A} \cap \rho^n = A \cap \rho^n \in N,$$

since $c \cup b <_* a = p_N^n$. Thus,

$$\mathbb{P}(\rho^n) \cap M \neq \mathbb{P}(\rho^n) \cap N,$$

contradiction! QED(Case 1)

Case 2. M_η is a proper segment of N_η.

Then M_η is sound. Hence M did not get moved in the iteration and $M = M_\eta$. But then N is not moved and $N = N_\eta, \eta = 0$, since otherwise ν_1 is a cardinal in N_η. But then $\lambda < \nu_1 \leq \mathrm{On}_M$ and $\rho^\omega_M \leq \lambda < \nu_1$, where M is a proper segment of N_η. Hence ν_1 is not a cardinal in N_η. Contradiction!

QED(Case 2)

Case 3. The above cases fail.

Then $M_\eta = N_\eta$ and I^N has a truncation on its main branch. We shall again prove: $M \in N$.

We first note the following:

Fact. Let Q. be acceptable. Let $\pi : Q \longrightarrow^*_F Q'$, where $\rho^{i+1} \leq \kappa < \rho^i$ in $Q, \kappa = \mathrm{crit}(F)$. Then:

$$\underline{\Sigma}^{(n)}_1(Q') \cap \mathbb{P}(\kappa) = \underline{\Sigma}^{(n)}_1(Q) \cap \mathbb{P}(\kappa) \text{ for } n \geq i.$$

Note. It follows easily that:

$$\underline{\Sigma}^{(n)}_1(Q') \cap \mathbb{P}(H) = \underline{\Sigma}^{(n)}_1(Q) \cap \mathbb{P}(H)$$

where $H = H^Q_\kappa = H^{Q'}_\kappa$.

We prove the fact. The direction $\supset$ is straightforward, so we prove $\subset$ by induction on $n \geq i$. The first case is $n = i$. Let $A \subset \kappa$ be $\Sigma^{(i)}_1(Q')$ in the parameter a. Then:

$$A_\xi \longleftrightarrow \bigvee z \in H^i_{Q'}\ B'(z, \xi, a)$$

where B' is $\Sigma^{(1)}_1(Q')$. But then π takes H'_Q cofinally to $H^i_{Q'}$. Hence:

$$A_\xi \longleftrightarrow \bigvee u \in H^{i'}_Q \bigvee z \in \pi(u) B'(\tau, \xi, a).$$

Let $a = \pi(f)\alpha$ where $f \in \Gamma^*(\kappa, Q)$ and $\alpha < \lambda(F) = F(\kappa)$. Let B be $\Sigma^{(i)}_0(Q)$ by the same definition as B'. Then:

$$A_\xi \longleftrightarrow \bigvee u \in H^i_Q \{\zeta < \kappa : \bigvee z \in u B(z, \xi, f(\alpha))\} \in F_\alpha,$$

where $F_\alpha \in \underline{\Sigma}_1(Q)$ by closeness.

This proves the case $n = i$. The induction step uses the fact that $\rho^n_Q = \rho^n_{Q'}$, for $n > i$. (Hence $H^n_Q = H^n_{Q'}$.)

Let $n = m + 1 > i$ and let it hold at m. Let $A \subset \kappa$ be $\underline{\Sigma}^{(m)}_1(Q')$. Then:

$$A_\xi \longleftrightarrow \langle H^n_{Q'}, B^1_\xi, \ldots, B^r_\xi\rangle \vdash \varphi$$

where φ is a Σ_1 sentence and:

$$B^h_\zeta = \{z \in H^n_Q : \langle \xi, z\rangle \in B^h\}\ (h = 1, \ldots, r)$$

and B^h is $\underline{\Sigma}^{(m)}_1(Q')$. We may assume w.l.o.g. that $B^h \subset H$. But then B^h is $\Sigma^{(m)}_1(Q)$. Hence A is $\underline{\Sigma}^{(n)}_1(Q)$.

QED(Fact)

Recall that $\rho^{l+1} \leq \lambda < \rho^l$ in M. Using this we get:

(1) There is a $\underline{\Sigma}^{(l)}_1(M)$ set $B \subset \lambda$ which codes M (in particular, if Q is a transitive ZFC^- model and $B \in Q$, then $M \in Q$.)

Proof. Recall from the definition of M that:

$$\overline{M} = M^{l,b} = h_{\overline{M}}(\omega \times (\lambda \times \{\overline{c}\})),\ \text{where}\ \overline{c} = \overline{b} \cap \rho^l_M.$$

Thus we can set:

$$\dot{M} = \{\prec i, \xi \in M : i < \omega, \xi < \lambda,\ \text{and}\ h_{\overline{M}}(i, \langle \xi, \overline{c}\rangle)\ \text{is defined}\}.$$

For $\prec i, \xi \succ \in \dot{M}$ set: $h(\prec i, \xi \succ) = h_{\overline{M}}(i, \prec \xi, \overline{c} \succ)$. Let $M = \langle J^E_\alpha, F\rangle$. We set:

- $\dot{\in} =: \{\langle x, y\rangle \in \dot{M}^2 : h(x) \in h(y)\}$
- $\dot{I} =: \{\langle x, y\rangle \in \dot{M}^2 : h(x) = h(y)\}$
- $\dot{E} =: \{x \in \dot{M} : h(x) \in E\}$
- $\dot{F} =: \{\ x \in \dot{M} : h(x) \in F\}$

Then:

$$\langle \dot{M}, \dot{\in}, \dot{E}, \dot{F}\rangle / I \cong \langle J^E_\alpha, F\rangle = M.$$

Let B be a simple coding of $\langle \dot{M}, \dot{\in}, \dot{E}, \dot{F}\rangle$, e.g. we could take it as the set of $\prec \xi, j \succ$ such that one of the following holds:

- $j = 0 \wedge \xi \dot{\in} \dot{M}$
- $j = 1 \wedge \xi = \prec \xi_u, \xi_1 \succ$ with $\xi_0 \dot{\in} \xi_1$
- $j = 2 \wedge \xi = \prec \xi_0, \xi_1 \succ$ with $\xi_0 I \xi_1$

- $j = 3 \wedge \xi \in \dot{E}$
- $j = 4 \wedge \xi \in \dot{F}$.

It is clear that if $B \in Q$ and Q is a transitive ZFC^- model, then $\overline{M}$ is recoverable from B in Q by absoluteness. Hence $\overline{M} \in Q$. But $\overline{M} = M^{l,\bar{b}}$ and M is recoverable from $\overline{M}$ in Q by absoluteness. Hence $M \in Q$.

QED(1)

Let $j+1$ be the final truncation point on the main branch of I^N. Then:

(2) B is $\underset{\sim}{\Sigma}_1^{(l)}(N_{j+1})$.

Proof. Let B be $\Sigma_1^{(l)}(M)$ in the parameter p. Let B' be $\Sigma_1^{(\theta)}(M_\eta)$ in $\pi(p)$ by the same definition, where $\pi = \pi^M_{1,\eta}$. Then $B = \lambda \cap B'$ is $\underset{\sim}{\Sigma}_1^{(l)}(N_\eta)$. Let i be the least $i \geq_T j+1$ in I^N set. B is $\Sigma_1^{(l)}(N_i)$. i is not a limit ordinal, since otherwise $\text{lub}\{\kappa_h : h \leq_{T^N} i\} = \text{lub}\{k_h : h < i\} > \lambda$ and there is $h \leq_{T^N} i$ such that $\kappa_h > \lambda$ and $a \in \text{rng}(\pi^N_{hi})$, where B is $\Sigma_1^{(l)}(N_i)$ in the parameter a. Hence B is $\underset{\sim}{\Sigma}_1^{(l)}(N_h)$. Contradiction! But then $i = k+1$. Let $t = T^N(k+1)$. If $k > j$, then $t \geq j+1$ and $\kappa_k \geq \lambda_j \geq \lambda > \rho_M^{l+1} = \rho_{N_\xi}^{l+1} = \rho_{N_t}^{l+1}$. By the above Fact we conclude that $B \in \underset{\sim}{\Sigma}_1^{(l)}(N_t)$ where $t < i$. Contradiction! Hence $i = j+1$. QED(2)

We consider two cases:

Case 3.1. $\kappa_j \geq \lambda$.

By the Fact, we conclude that B is $\underset{\sim}{\Sigma}_1^{(i)}(N_j^*)$ is a proper segment of N_t, where $t = T^N(j+1)$. Hence $B \in \underset{\sim}{\Sigma}_1^{(i)}(N_j^*) \subset N$. But then $B \cap \mathbb{P}(\lambda) \cap N \subset J^{E^N}_{\sigma(\lambda)}$, since $\sigma(\lambda) > \lambda$ is regular in N. Hence $J^{E^N}_{\sigma(\lambda)}$ is a ZFC^- model and $M \in J^{E^N}_{\sigma(N)} \subset N$.

QED(Case 3.1)

Case 3.2. Case 3.1 fails.

Then $\kappa_j < \lambda$. But $\tau_j \geq \lambda$, since otherwise $\tau_j < \lambda$ is a cardinal in M, hence in N. Hence $N_j^* = N$ and no truncation would take place at $j+1$. Contradiction! Thus:

$$\lambda = \tau =: \tau_j,\ N_j^* = N^* = N||\gamma,\ \kappa_j = \kappa,$$

where κ is the cardinal predecesor of λ in M and $\gamma > \lambda$ is maximal such that τ is a cardinal in $N||\gamma$. Then:

(1) $\pi : N^* \longrightarrow^*_F N_{j+1}$ where $\pi = \pi^N_{0,j+1}$, $F = E^{N_j}_{\nu_j}$

Since:

$$\pi_{j+1,\eta} : N_{j+1} \longrightarrow_{\Sigma^*} M_\eta \text{ and } \operatorname{crit}(\pi_{j+1,\eta}) > \lambda,$$

we know that:

(2) $\rho^{l+1} < \lambda < \rho^l$ in N_{j+1}

By the definition of N^* we have: $\rho^\omega_{N^*} < \lambda$. But $\rho^\omega_{N^*} \geq \kappa$, since κ is a cardinal in N and $N^* \in N$. Hence:

(3) $\rho^\omega_{N^*} = \kappa$.

Now let: $\rho^{i+1} \leq \kappa < \rho^i$ in N^*. Then:

$$\rho^{i+1} \leq \kappa < \lambda \leq \rho^i \text{ in } N_{j+1},$$

since:

$$\lambda < \sup \pi\text{''}\lambda = \lambda(F) \leq \sup \pi\text{''}\rho^i_{N^*} = \rho^i_{N_{j+1}}.$$

Hence $i = l$ and:

(4) $\rho^{l+1} = \kappa < \rho^l$ in N_{j+1}.

We now claim:

(5) $B \in \operatorname{Def}(N^*)$, i.e. B is definable in parameters from N^*.

Proof. For $\xi < \lambda$ define a map $g_\xi : \kappa \longrightarrow \kappa$ as follows:

For $\alpha < \kappa$ set:

- $X_\alpha =$ the smallest $X \prec J^{E^{N^*}}_\lambda$ such that $\alpha \cup \{\xi\} \in X$.
- $C_\xi = \{\alpha < \kappa : X_\xi \circ k \subset \alpha\}$.

For $\alpha \in C_\xi$, let $\sigma_\xi : Q_\xi \overset{\sim}{\longleftrightarrow} X_\xi$ be the transitivator of X_ξ. Set:

$$g_\xi(\alpha) =: \begin{cases} \sigma_\xi^{-1}(\xi) & \text{if } \alpha \in C_\xi \\ \varnothing & \text{if not} \end{cases}$$

It is easily seen that:

$$\pi(g_\xi)(\kappa) = \xi \text{ where } \pi = \pi^N_{0,j+1}.$$

Since B is $\underset{\sim}{\Sigma}^{(l)}_1(N_{j+1})$ we have:

$$B_\zeta \longleftrightarrow \bigvee u \in J^{E^{N^v}}_{\rho_{N^*}} \bigvee z \in \pi(u) B'(z, \zeta, u).$$

Let $f \in \Gamma^*(\kappa, N^*)$ such that $a = \pi(f)(\alpha), \alpha < \lambda$. We know that $\xi = \pi(g_\xi)(k)$ for $\xi < \lambda$. But then the statement B_ζ is equivalent to

$$\bigvee u \in J^{E^{N^v}}_{\rho^f_{N^*}} \{\langle \mu, \delta\rangle : \bigvee x \in u B''(x, g_\zeta(\mu), f(\delta))\} \in F_{\langle K, \alpha\rangle}$$

where $F = E^{N_j}_{\nu_j}$ and B'' is $\Sigma^{(l)}_0(N^*)$ by the same definition. But $F_{\langle \kappa, \alpha\rangle}$ is $\underset{\sim}{\Sigma}_1(N^*)$ by closeness. QED(5)

But then $B \in \operatorname{Def}(N^*) \subset J^{E^N}_{\sigma(\lambda)} \subset N$. Hence $M \in N$.

QED(Lemma 4.3.1)

4.3.2 Soundness and Cores

Let N be any acceptable structure. Let $m < \omega$. In §2.5 we defined the set R^n_N of *very* good n-parameters. The definition is equivalent to:

> $a \in R^n$ iff a is a finite set of ordinals and for $i < n$, each $x \in N||\rho^i$ has the form $F(\xi, a)$ where F is a $\Sigma_1^{(i)}(N)$ map and $\xi < \rho^{i+1}$.

We said that N is n-sound iff $R^n_N = P^n_N$. It follows easily that N is n-sound iff $p^n \in R^n$, where $p^n = p^n_N$ is the $<_*$-least $p \in P^n$. We called N **sound** iff it is n-sound for all n. It followed that, if N is sound, then $\rho^n \setminus \rho^i = p^i$ for $i \leq n < \omega$.

We have now shown that, if N is a mouse then $p^n \setminus \rho^i = p^i$ for $i \leq n < \omega$, regardless of soundness. We set: $p^* = \bigcup_{n<\omega} p^n$. Then $p^* = p^n$ whenever $\rho^n = \rho^\omega$ in N. We know:

Lemma 4.3.2. *If N is a mouse and $\pi : \overline{N} \longrightarrow_{\Sigma^*} N$ strongly, then $\overline{N}$ is a mouse and $\pi(p^*_{\overline{N}}) = p^*_{N^*}$.*

Proof. $\overline{N}$ is a mouse by a copying argument. Hence $\overline{N}$ is solid. But then $\pi(p^i_{\overline{N}}) = P^i_N$ for all $i < \omega$, by Lemma 4.1.11.

QED(Lemma 4.3.2)

We know generalize the notion R^n_N as follows:

Definition 4.3.1. Let $\rho^\omega_N \leq \mu \in N, a \in R^{(\mu)}_N$ iff a is aa finite set of ordinals and for some n,

- $\rho^n \leq \mu < \rho^{n-1}$ in N.
- Every $x \in N||\rho^{n-1}$ has the form $F(\vec{\xi}, a)$, where $\xi_1, \ldots, \xi_r < \mu$ and F is $\Sigma_1^{(n-1)}(N)$.
- If $j > n - 1$, then $a \in R^j_N$.

We also set:

Definition 4.3.2. N is sound above μ iff for some $n, \rho^n \leq \mu < \rho^{n-1}$ in N and whenever $p \in P^n_N$ then $p \setminus \mu \in R^{(\mu)}_N$.

(It again follows that N is sound above μ iff $p^n_N \setminus \mu \in R^{(\mu)}_N$.) We prove:

Lemma 4.3.3. *Let N be a mouse. Let $\rho^\omega_N \leq \mu \in N$. There is a unique pair σ, M such that:*

- $\sigma : M \longrightarrow_{\Sigma^*} N$
- *M is a mouse which is sound above μ*
- $\sigma \upharpoonright \mu = \mathrm{id}$ *and* $\sigma(p^*_M) = p^*_N$.

Before proving this, we develop some of its consequences.

Definition 4.3.3. Let N be a mouse. If M, σ are as above, we call M the **μ-th core of** N, denoted by: $\mathrm{core}(N) = \mathrm{core}_{\rho^\omega_N}(N)$, and σ the **μ-th core map**, denoted by σ^N_μ.

We also set: $\mathrm{core}(N) = \mathrm{core}_{\rho^\omega_N}(N)$ and $\sigma^N = \sigma^N_{\rho^\omega_N}$, $M = \mathrm{core}(N)$ is the **core** of N, and σ^N is the **core map**.

We leave it to the reader to prove:

Corollary 4.3.4. *Let N be a mouse. Then:*

- $\mathrm{core}_\mu(\mathrm{core}_\mu(N)) = \mathrm{core}_\mu(N)$.
- *N is sound above μ iff $N = \mathrm{core}_\mu(N)$.*
- *Let $M = \mathrm{core}_\mu(N), \overline{\mu} \leq \mu, \overline{M} = \mathrm{core}_{\overline{\mu}}(M)$.*

Then $\overline{M} = \mathrm{core}_{\overline{\mu}}(M)$ and $\sigma^N_\mu \sigma^M_{\overline{\mu}} = \sigma^N_{\overline{\mu}}$.

We now turn to the proof of Lemma 4.3.3. By Löwenheim-Skolem argument it suffices to prove it for countable N. We first prove uniqueness. Suppose not. Let M, π and M', π' both have the property. If $x \in M$, then $x = F(\vec{\xi}, P^*_N)$ where F is good and $\xi_1, \ldots, \xi_r < \mu$, since M is sound above μ. Hence:

$$\pi(x) = \tilde{F}(\vec{\xi}, P^*_N)$$

where $\tilde{F}$ has the same good definition over N. But then in N the Σ^* statement holds:

$$\bigvee y\, y = \tilde{F}(\vec{\xi}, P^*_N).$$

(This is Σ^* since it results from the substitution of $\tilde{F}(\vec{\xi}, P^*_N)$ in the formula $\nu = \nu$.) Hence in M' we have:

$$\bigvee y\, y = F'(\vec{\xi}, P^*_N),$$

where F' has the same good definition over M'. Thus $\text{rng}(\pi) \subset \text{rng}\,\pi'^{-1}$ and $\pi'^{-1}\pi$ is a Σ^*-preserving map of M to M'. A repeat of this argument then shows that $\text{rng}(\pi') \subset \text{rng}(\pi^{-1})$ and $\pi'^{-1}\pi$ is an isomorphism of M onto M'. But M, M' are transitive. Hence $M = M'$ and $\pi = \pi'$.

QED

This prove uniqueness. We now prove existence. Let $a = p^*_N$. Let $\rho^{n+1} \leq \mu < \rho^n$. Set $\overline{N} = N^{n,a}$. Let $b = a \cap \rho^n_N$ and set:

$$X = h_{\overline{N}}(\mu \cup b) = \text{ the closure of } \mu \cup b \text{ under } \Sigma_1(\overline{N}) \text{ functions.}$$

Let $\overline{\sigma} : \overline{M} \overset{\sim}{\longleftrightarrow} \overline{N}|X$ be the transitivazation of $\overline{N}|X$. By the downward extension lemma, there are unique $M, \sigma \supset \overline{\sigma}, \overline{a}$ such that:

$$\overline{M} = M^{n,\overline{a}},\ \sigma : M \longrightarrow_{\Sigma_1^{(n)}} N,\ \sigma(\overline{a}) = a.$$

Clearly, $\sigma \restriction \mu = \text{id}$. Moreover, $\overline{a} \in R^{(\mu)}_M$. It suffices to prove:

Claim. σ is Σ^*-preserving and $\overline{a} = p^*_M$.

If $\sigma = \text{id}$ and $M = N$, there is nothing to prove, so suppose not. Let $\lambda = \text{crit}(\sigma)$. (Hence $\mu \leq \lambda$.) There is then a $h \leq n$ such that $\rho^{h+1} \leq \lambda < \rho^h$ in N. λ is a regular cardinal in M, since $\sigma(\lambda) > \lambda$. It follows easily that σ witnesses the phalanx $\langle N, M, \lambda \rangle$. Note that $\rho^\omega_M \leq \mu \leq \lambda$, since $\overline{a} \in R^{(\mu)}_M$. We now apply the simplicity lemma, coiterating $N, \langle N, M\lambda \rangle$ with:

$$I^N = \langle \langle N_i \rangle, \langle \nu^N_i \rangle, \langle \pi^N_{i,j} \rangle, T^N \rangle$$

$$I^M = \langle \langle M_i \rangle, \langle \nu^M_i \rangle, \langle \pi^M_{i,j} \rangle, T^M \rangle$$

being the iteration of $N, \langle N, M, \lambda \rangle$ respectively. We assume that the iteration terminates at an $\eta < \omega_1$ and that $\langle \nu_i : 1 \leq i < \eta \rangle$ is the sequence of coindices.

It is now time to mention that some of the steps in the proof of solidity go through with a much weaker assumption on the phalanx $\langle N, M, \lambda \rangle$ and its witness σ. In particular:

Lemma 4.3.5. *Let σ witness $\langle N, M, \lambda \rangle$, where $R^{(\lambda)}_M \neq \varnothing$. If cases 2 or 3 hold, then $M \in N$.*

The reader can convince himself of this by an examination of the solidity proof. But the premises of Lemma 4.3.5 is given. Hence:

(1) Case 1 applies.

Proof. Suppose not. Let A be $\Sigma_1^{(h)}(N)$ in a such that $A \cap \rho_N^{h+1} \notin N$. Let $\overline{A}$ be $\Sigma_1^{(h)}(M)$ in $\overline{a}$ by the same definition. Then $A \cap \rho_N^{h+1} = \overline{A} \cap \rho_N^{h+1} \in N$, since $\overline{A} \in \underline{\Sigma}_\omega(M) \subset N$. Contradiction!

QED(1)

Then $M_\eta = N_\eta$ and there is no truncation on the main branch of I^N. Then $\pi_{1,\eta}^M : M \longrightarrow_{\Sigma^*} M_\eta$. Hence, by a copying argument, M is a mouse, hence is solid. Since $\mathrm{crit}(\pi_{1,\eta}^M) \geq \lambda$, we have:

(2) $\mathbb{P}(\lambda) \cap M = \mathbb{P}(\lambda) \cap M_\eta$ and $\rho_M^i = \rho_{M_\eta}^i$ for $i > h$.

But:

(3) $\mathrm{crit}(\pi_{1,\eta}^N) \geq \rho^{h+1}$.

Proof. Suppose not. then there is $j+1 \leq_{T^N} \eta$ such that $\kappa_j < \rho^{h+1}$. Let j be the least such. Let $t = T^N(j+1)$. Then:

$$\kappa_j < \sup\ \pi_{t,j+1}"\rho_N^{h+1} \leq \rho_{N_{j+1}}^{h+1} \leq \rho_{N_\eta}^{h+1} = \rho_M^{h+1} > \kappa_j.$$

Contradiction!

QED(3)

Hence:

(4) $\rho_N^i = \rho_M^i$ for $i > h$. Moreover if $\rho^i = \rho_N^i$, then $\mathbb{P}(\rho^i) \cap N = \mathbb{P}(\rho^i) \cap M$ for $i > h$.

Using this we get:

(5) $\sigma : M \longrightarrow_{\Sigma^*} N$.

We first show that σ is Σ^*-preserving. By induction on $i \geq h$ we show:

Claim. σ is $\Sigma_1^{(i)}$-preserving.

For $i = h$, this is given. Now let $i = k+1 \geq h$ and let it hold for k. Let A be $\Sigma_1^{(i)}(M)$. then:

$$Ax \longleftrightarrow \langle H^i, B_x^1, \ldots, B_x^r \rangle \models \varphi$$

where φ is a Σ_1-sentence and:

$$B_x^i \{z \in H^i : \langle z, x \rangle \in B^l\},$$

where B^l is $\Sigma_1^{(k)}(M)$ for $l = 1, \ldots, r$. Let A' be $\Sigma_1^{(k)}(M)$ by the same definition. Then:

$$B_{zx}^l \longleftrightarrow B_{z\sigma(x)}^{l'} \text{ for } z \in H_M^i = H_N^i.$$

Hence $Ax \longleftrightarrow A'\sigma(x)$.

QED(5)

But

(6) σ is strongly Σ^*-preserving.

Proof. Let $\rho^m = \rho^\omega$ in M and N. Let A be $\Sigma_1^{(m)}(M)$ in x such that $A \cap \rho^m \notin M$. Let A' be $\Sigma_1^{(m)}(M)$ in $\sigma(x)$ by the same definition. Then $A \cap \rho^n = A' \cap \rho^m \notin N$, since $\mathbb{P}(\rho^m) \cap M = \mathbb{P}(\rho^m) \cap N$.

QED(6)

But then $\sigma(P_M^*) = P_N^*$. Hence $P_M^* = \overline{a} = \overline{\sigma}'(P_N^*)$. We know that $\overline{a} \in R_M^{(\mu)}$. Hence M is solid above μ.

QED(Lemma 4.3.5)

4.3.3 Condensation

The condensation lemma for L says that if M is transitive and $\pi : M \longrightarrow J_\alpha$ is a reasonable embedding, then $M \triangleleft J_\alpha$. It is natural to ask whether the dame holds when we replace J_α by an arbitrary sound mouse. In order to have any hope of doing this, we must employ a more restrictive notion of reasonable. Let us call $\sigma : M \longrightarrow N$ reasonable iff either $\sigma = \text{id}$ or σ witnesses the phalanx $\langle N, M, \lambda \rangle$ and $\rho_M^\omega \leq \lambda$. We then get:

Lemma 4.3.6. *If N, M are sound mice and $\sigma : M \longrightarrow N$ is reasonable in the above sense, then $M \triangleleft N$.*

It ifs not too hard to prove this directly from the solidity lemma and the simplicity lemma. We shall, however, derive it from a deeper structural lemma:

Lemma 4.3.7. *Let N be a mouse. Let σ witness the phalanx $\langle N, M, \lambda \rangle$. Then M is a mouse. Moreover, if M is sound above λ, then one of the following hold:*

(a) $M = \text{core}_\lambda(N)$ *and* $\sigma = \sigma_\lambda^N$.

(b) M *is a proper segment of* N.

(c) $\pi : N||\gamma \longrightarrow_F^* M$, *where* $F = F_\mu^N$ *such that:*

  (i) $\lambda < \gamma \in N$ *such that* $\rho_{N||\gamma}^\omega < \lambda$.

(ii) $\lambda = \kappa^{+N||\gamma}$ *where* $\kappa = \mathrm{crit}(F)$.

(iii) F *is generated by* $\{\kappa\}$.

Remark. In case (c) we say that M is one measure away from N. Then γ is maximal such that λ is a cardinal in $N||\gamma$. Hence $\rho_{N||\gamma} \leq \kappa$. But κ is a cardinal in N and $N||\gamma \in N$. Hence $\rho_{N||\gamma} = \kappa$. But $\pi \restriction \kappa = \mathrm{id}$ and $\pi(p^*_{N|\gamma}) = p^*_M$. Hence $N||\gamma = \mathrm{core}(M)$ and π is the core map. Clearly, μ is least such that $E^M_\mu \neq E^N_\mu$.

Remark. Lemma 4.3.6 follows easily, since the possibilities (a) and (c) can be excluded. (a) cannot hold, since otherwise $M = \mathrm{core}_\lambda(N) = N$ by the soundness of N. Hence $\sigma^\lambda_N = \mathrm{id}$. Contradiction, since $\mathrm{crit}(\sigma^\lambda_N) = \lambda$. If (c) held, then $N^* = \mathrm{core}(M)$ where $N^* = N||\gamma$, and π is the core map. But M is sound. Hence $M = N^* = \mathrm{core}(M)$ and $\pi = \mathrm{id}$. Contradiction!

Remark. Lemma 4.3.7 has many applications, through mainly in setting where the awkward possibility (c) can be excluded (e.g. when λ is a limit cardinal in M). We have given a detailed description of (c) in order to facilitate such exclusions.

We now prove Lemma 4.3.7. We can again assume N to be countable by Löwenheim-Skolem argument. We again coiterate against $\langle N, M, \lambda\rangle$ getting the iterations:

$$I^N = \langle\langle N_i\rangle, \dots, T^N\rangle,\ I^M = \langle\langle M_i\rangle, \dots, T^M\rangle$$

with coiteration indices $\langle \nu_i : i < \eta\rangle$, where the coiteration terminates at $\eta < \omega_1$. Then $\pi_{1,\eta} : M \longrightarrow_{\Sigma^*} M_\eta$ and M is a mouse by a copying argument. Now let M be sound above λ. We again consider three cases:

Case 1. $M_\eta = N_\eta$ and I^N has no truncation on the main branch.

We can literally repeat the proof in cases of Lemma 4.3.5, getting:

$$\sigma \text{ is strongly } \Sigma^*\text{-preserving.}$$

Hence $\sigma(p^*_M) = p^*_N$ where M is sound above λ and $\sigma = \sigma^N_\lambda$.

QED(Case 1)

Case 2. M_η is a proper segment of N_η.

We can literally repeat the proof in Case 2 of the solidity Lemma, getting: M is a proper segment of N.

Case 3. The above cases fail.

Then $M_\eta = N_\eta$ and I^N has a truncation on the main branch. Let $j+1$ be the last truncation point on the main branch. Then M is a mouse and $\pi^M_{1,\eta}$ is strongly Σ^*-preserving. Hence $\pi^M_{1,\eta}(p^*_M) = p^*_{M_\xi}$. But $\kappa_i \geq \lambda$ for all $i \leq_{T^M} \eta$. Hence $\text{crit}(\pi_{1,\eta}) \geq \lambda$. Hence:

$$M = \text{core}_\lambda(M_\eta) \text{ and } \pi_{1,\eta} = \sigma_\lambda^{M_\xi},$$

since M is sound above λ. We also know:

$$\kappa_i \geq \lambda_j \geq \lambda \text{ for } j+1 <_{T^N} i+1 <_{T^N} \eta.$$

Hence $\text{crit}(\pi^N_{j+1,\eta}) \geq \lambda$ and $\pi^N_{j+1,\eta}(P^*_{N_{j+1}}) = p^*_{N_\eta} = p^*_{M_\eta}$. Hence:

$$M = \text{core}_\lambda(N_{j+1}) \text{ and } \sigma_\lambda^{N_{j+1}} = (\pi^N_{j+1,\eta})^{-1} \circ \pi^M_{1,\eta}.$$

We consider two cases:

Case 3.1. $\kappa_j \geq \lambda$.

Then N^*_j is a proper initial segment of N_j, hence is sound. Since $\kappa_j \geq \lambda$, it follows as before that $M = \text{core}_\lambda(N^*)$. Hence $M = N^*_j$ by the soundness of N^*_j. But this means that M was not moved in the iteration I^M up to $t = T^N(j+1)$, since if $h < t$ in the least point active in I^*, then $E^M_{\nu_h} \neq \varnothing$ and hence $E^{N_t}_{\nu_h} = E^{N^*_j}_{\nu_h} = \varnothing$. Hence $N^*_j \neq M$. Contradiction!

Thus $M_t = M = N^*_j$ is a proper segment of N_t. Hence the coiteration terminates at $t < \eta$. Contradiction!

QED(Case 3.1)

Case 3.2. Case 3.1 fails.

Then $\kappa_j < \lambda$. But $\tau_j \geq \lambda$, since otherwise τ_j is a cardinal in N and $N^*_j = N$. Hence $j+1$ is not a truncation point in I^N. Contradiction!

Thus $\tau_j = \lambda$. $t = T^N(j+1)$ is the least i which is active in I^N (since $\kappa_j < \lambda \leq \lambda_i$). But then $N = N_t$ and $N^*_j = N^* = N||\gamma$, where γ is maximal such that $\tau = \lambda$ is a cardinal in $N||\gamma$. Hence $\kappa_j = \kappa =$ the cardinal predecesor of τ in N^*. $\kappa = \rho^\omega_{N^*}$, since κ is a cardinal in N and $N^* \in N$. We have:

$$\kappa_i \geq \lambda \text{ for } 1 \leq_{T^M} i+1 \leq_{T^M} \eta$$

Hence $\text{crit}(\pi^M_{1,\eta}) \geq \lambda$. But:

$$\kappa_i \geq \lambda_t \geq \lambda \text{ for } j+1 <_{T^N} i+1 <_{T^N} \eta$$

Hence $\text{crit}(\pi^N_{j+1,\eta}) \geq \lambda$. Hence:

$$M = \text{core}_\lambda(N_{j+1}),\ (\pi^N_{j+1,\eta})^{-1} \circ \pi^M_{1,\eta} = \sigma^{N_{j+1}}_\lambda,$$

$\rho^\omega_{N^*} \leq \kappa$. But then $\rho^\omega_{N^*} = \kappa$ since κ is a cardinal in N and $N^* \in N$. Set $\langle \tilde{N}, \tilde{F} \rangle = N_j || \nu_j$. Then:

$$\pi_{t,j+1} : N^*_j \longrightarrow^*_{\tilde{F}} N_{j+1}$$

By closeness we have: $\tilde{F}_\kappa \in \underline{\Sigma}_1(N^*)$. Hence $\tilde{F}_\kappa \in \underline{\Sigma}_1(N^*) \subset N||\sigma(\tau)$, where $\sigma(\tau)$ is regular in N and $\gamma < \sigma(\tau)$. By a standard construction there is a unique premouse $\langle Q, F \rangle$ such that $F_\kappa = \tilde{F}_\kappa, Q||\tau = N||\tau$ and F is generated by $\{\kappa\}$. To see this we have:

$$\tilde{F}|\kappa + 1(X) = \begin{cases} X \cup \{\kappa\} & \text{if } X \in \tilde{F}_\kappa \\ X & \text{if not} \end{cases}$$

Then $\tilde{F}|\kappa + 1 \in N||\sigma(\tau)$. Then $\langle Q, F \rangle$ is the extension of $\langle N|\tau, \tilde{F}|\kappa + 1 \rangle$ in the sense of §3.2. By a standard construction there exist:

$$\pi : N^* \longrightarrow^*_F M',\ \sigma' : M' \longrightarrow_{\Sigma^*} N_{j+1}$$

such that $\sigma'(\pi(f)(\kappa)) = \pi_{t,j+1}(f)(\kappa)$ for $f \in \Gamma^*(\kappa, N^*)$.

To see this, let $\overline{\pi} : N||\tau \longrightarrow Q$ be the extension map. Then each $\alpha < \lambda_t$ has the form $\overline{\pi}(f)(\kappa)$ for an $f \in N||\tau$. Set: $g(\alpha) = \pi_{t,j+1}(f)(\kappa)$. Then $g : \lambda_F \longrightarrow \lambda_j$ and:

$$\langle \text{id}, g \rangle : \langle N||\tau, F \rangle \longrightarrow^* \langle \tilde{N}, \tilde{F} \rangle$$

as defined in §3.2. Hence, the are π, M', σ' as above with:

$$\sigma'(\pi(f)(\alpha)) = \pi_{t,j+1}(f)(g(\alpha)) \text{ for } \alpha < \lambda_{F'}.$$

Then $\sigma' \upharpoonright \tau = \text{id}$ and:

$$\sigma'(p^*_{M'}) = \sigma'\pi(p^*_{N^*}) = \pi_{t,j+1}(p^*_{N^*}) = p^*_{N_{j+1}}.$$

Hence $M = \text{core}_\tau(M')$ and $\sigma^{M'}_\tau = (\sigma')^{-1} \circ \sigma^\tau_{N_{j+1}}$. However:

Claim 1. M' is sound above τ. Hence $M = M' = \text{core}_\tau(N_{j+1})$.

Proof. Let $\rho^n \leq \kappa \leq p^{n-1}$ in N^*. Hence $\kappa = \rho^n = \rho^\omega$ in N^*. Let $x \in M'$. Then $x = \pi(f)(\kappa)$, where $f \in \Gamma^*(\kappa, N^*)$.

By the soundness of N^* we may assume:

$$f(\xi) = F(\xi, a, \vec{\zeta})$$

where F is a good $\Sigma_1^{(n-1)}(N^*)$ function, $a = p^n_{N^*}$ and $\zeta_1, \ldots, \zeta_r < \kappa$. Hence:

$$\pi(f)(\kappa) = F'(\kappa, \pi(a), \vec{\zeta})$$

where F' is $\Sigma_1^{(n-1)}(M')$ by the same good definition, $\pi(a) = p^n_M$, and $\vec{\zeta} < \tau$. But $\kappa < \tau$, where $\rho^n < \tau < \rho^{n-1}$ in M'.

QED(Claim 1)

All that remains is to show:

Claim 2. $\langle Q, F\rangle = N||\mu$ for a $\mu \leq \gamma$.

Proof. We note that if $\langle Q, F\rangle = N||\mu$, then we automatically have $\mu \leq \gamma$, since τ is then a cardinal in $N||\mu$ and γ is maximal s.t. τ is a cardinal in $N||\gamma$.

(1) $\langle Q, F\rangle \in N$.

Proof.$(E^{N_\gamma}_{\nu_j})_\kappa = F_\kappa \in N||\sigma(\tau)$, where $N||\sigma(\tau)$ is a ZFC^- model. Hence $\langle Q, F\rangle \in N||\sigma(\tau)$ since the construction of $\langle Q, F\rangle$ can be carried out in $N||\sigma(\tau)$ by absoluteness.

(2) $\rho^1_{\langle Q,F\rangle} \leq \tau$.

Proof. As above, let $\overline{\pi} : N||\sigma(\tau) \longrightarrow Q$ be the extension map given by F. By §3.2 we know that $\overline{\pi}$ is $\underline{\Sigma}_1(\langle Q, F\rangle)$ and that $\langle Q, F\rangle$ is amenable. But then there is a $\underline{\Sigma}_1(\langle a, \pi\rangle)$ partial map G of $N||\tau$ onto Q defined by: $G(f) = \overline{\pi}(f)(\kappa)$ for $f \in N||\tau, : f : \kappa \longrightarrow N||\tau$.

QED(2)

Define a map $\tilde{\sigma} : \langle Q, F\rangle \longrightarrow N_j||\nu_j$ by:

$$\tilde{\sigma}(\overline{\pi}(f)(\kappa)) := \tilde{\pi}(f)(\kappa) \text{ for } f \in N|\tau, f : \kappa \longrightarrow N||\tau,$$

where $\tilde{\pi} = \pi^N_{t,i} \upharpoonright (N||\tau)$ is the extension of $N_j||\nu_j$.

Then:

(3) $\tilde{\sigma} : \langle Q, F\rangle \longrightarrow_{\Sigma_0} N_j||\nu_j$. In fact, it is also cofinal.

(4) $\tilde{\sigma} \upharpoonright \tau + 1 = \text{id}$.

Proof. Set:

$$i^+ =: \text{ the least } \eta > i \text{ such that } \eta = \overline{\overline{\eta}} \geq \omega \text{ in } Q$$
$$pl := \langle i^+ : i < \kappa\rangle.$$

Then $\overline{\pi}(pl)(\kappa) = \kappa^{+Q} = \kappa^{+N_j||\nu_j} = \tilde{\pi}(pl)(\kappa)$.

Set:

$$\Gamma =: \{f \in N||\tau : f : \kappa \longrightarrow \kappa \wedge f(i) < i^{+} \text{ for } i < \kappa\}$$
$$\dot{<} = \{\langle f, g\rangle \in \Gamma : \{i : f(i) \in g(i)\} \in F_\kappa\}$$

Then every $\xi < \tau$ has the form $\overline{\pi}(f)(\kappa)$ fo an $f \in \Gamma$. Clearly, $f \dot{<} g \longleftrightarrow \overline{\pi}(f)(a) < \pi(g)(a)$ for $f, g \in \Gamma$. Hence by $\dot{<}$-induction on $g \in \Gamma$:

$$\pi(g)(\kappa) = \{\overline{\pi}(\kappa) : f \dot{<} g\}.$$

But $F_\kappa = (E^{N_j}_{\nu_j})_\kappa$. Hence the same holds for $\tilde{\pi}$ in place of $\overline{\pi}$. Thus, by $\dot{<}$-induction on $g \in \Gamma$:

$$\tilde{\pi}(g)(\kappa) = \{\tilde{\pi}(\kappa) : f \dot{<} g\} = \{\pi(\kappa) : f \dot{<} g\} = \overline{\pi}(f)(\kappa).$$

Hence $\tilde{\sigma} \upharpoonright \tau = \text{id}$. But:

$$\tilde{\sigma}(\tau) = \tilde{\sigma}(\overline{\pi}(pl)(\kappa)) = \overline{\pi}(pl)(\kappa) = \tau$$

QED(4)

Redoing the proof of (2) with more care, we get:

(5) $\varnothing \in R^{(\tau)}_{\langle Q,F\rangle}$.

Proof. $X \subset \kappa$ and $X = \kappa$ are both $\Sigma_1(\langle Q, F\rangle)$, since:

$$X \subset \kappa \longleftrightarrow X \in \text{dom}(F),\ X = \kappa \longleftrightarrow X \in \text{On} \cap \text{dom}(F).$$

Thus this suffices to show that $\overline{\pi}$ is $\Sigma_1(\langle Q, F\rangle)$. We note that if $f : X \overset{\text{onto}}{\longrightarrow} u$ and u is transitive, then $\overline{\pi}(f) : \overline{\pi}(X) \overset{\text{onto}}{\longrightarrow} \overline{\pi}(u)$ and $\overline{\pi}(u)$ is transitive. But $\overline{\pi}(X) = F(X)$ for $X \subset \kappa$. Hence $y = \overline{\pi}(x)$ can be expressed by saying that there are:

$$X, Y, f, u, X', Y', f', u'$$

such that:

$$\bigvee u \wedge X, Y \in \text{dom}(F) \wedge f : X \overset{\text{onto}}{\longrightarrow} u \wedge x = f(0)$$
$$\wedge \bigwedge \xi, \zeta \in X(f(\xi) \in f(\zeta) \longleftrightarrow \prec \xi, \zeta \succ \in Y)$$
$$\wedge X' = F(X) \wedge Y' = F(Y) \wedge f' : X' \overset{\text{onto}}{\longrightarrow} u' \wedge y = f'(0)$$
$$\wedge \bigwedge \xi, \zeta \in X'(f'(\xi) \in f'(\zeta) \longleftrightarrow \prec \xi, \zeta \succ \in Y')$$

QED(5)

We then prove:

(6) One of the following holds:

(a) $\langle Q, F\rangle = \text{core}_\tau(N_j||\nu_j)$ and $\tilde{\sigma}$ is the core map.

(b) $\langle Q, F\rangle$ is a proper segment of $N_j||\nu_j$

(c) $\rho^\omega > \tau$ in $\langle Q, F\rangle$.

Proof. If $\tilde{\sigma} = \text{id}, \langle Q, F\rangle = N_j||\nu_j$, then (a) holds. Now let $\tilde{\sigma} \neq \text{id}$. Let $\tilde{\lambda} = \text{crit}(\tilde{\sigma})$. Then $\tilde{\lambda} > \tau$ by (4). We know $\rho^1 \leq \tau \leq \tilde{\lambda}$ in $\langle Q, F\rangle$. Moreover $\tilde{\sigma}$ is Σ_0-preserving. It follows easily that $\tilde{\sigma}$ verifies the phalanx $\langle N_j||\nu_j, \langle Q, F\rangle, \tilde{\lambda}\rangle$. $\langle Q, F\rangle$ is then a mouse. Moreover, it is sound above τ since $\varnothing \notin R^{(\sigma)}_{\langle Q,F\rangle}$. Hence it is sound above $\tilde{\lambda}$ since $\tau < \tilde{\lambda}$. We then coiterate $N_j||\nu_j$ against $\langle N_j||\nu_j, \langle Q, F\rangle, \tilde{\lambda}\rangle$, using all what we have learned up until now. We consider the same three cases. In case 1, (a) holds. In case 2, (b) holds. We now consider case 3, using what we have learned up to now. We know that $\tilde{\lambda}$ is a successor cardinal in $\langle Q, F\rangle$ and that its predecessor $\tilde{\kappa}$ is a limit cardinal in $\langle Q, F\rangle$. Since $\tau < \tilde{\lambda}$ is a successor cardinal in $\langle Q, F\rangle$, we conclude: $\tau < \tilde{\kappa} = \rho^\omega$.

(7) $\langle Q, F\rangle$ is a proper segment of N.

Proof. Suppose not. We derive a contradiction. (c) cannot hold, since $\rho^\omega \leq \tau$ in $\langle Q, F\rangle$. Now let (b) holds. Then $\langle Q, F\rangle$ is a proper segment of N_j. Hence $N_j \neq N$. Hence there is a least $i < j$ which is active in I^N. Thus $J^{E^N}_{\nu_i} = J^{E^{N_\gamma}}_{\nu_i}$ where $\nu_i > \tau$ is regular in N_j. But $\rho^\omega_{\langle Q,F\rangle} \leq \tau$.

Hence $\text{card}(\langle Q, F\rangle) \leq \tau$ in N_j and $\langle Q, F\rangle$ is a proper segment of $J^{E^N}_{\nu_i}$. Contradiction!

Now let (a) hold. If $\nu_j \in N_j$, then $N_j||\nu_j$ is sound. Hence $\tilde{\sigma} = \text{id}$, $\langle Q, F\rangle = N_j||\nu_j$. Hence $\langle Q, F\rangle$ is a proper segment of N_j and we can argue as above. Contradiction!

Now let $\nu_j \notin N_j$. Then $N_j = N_j||\nu_j = \langle J^{E^{N_j}}_{\nu_j}, {E^{N_j}}_{\nu_j}\rangle$. We now show:

Claim. Let $h = T^N(i+1)$ where $i + 1 \leq_{T^N} j$ and $(i+1, j]_{T^N}$ is truncation free. Then $\kappa_i > \tau$.

Proof. Suppose not. Recall that $\tau = \lambda_0 < \lambda_i$ for $i > 0$.

We have $\tau = \tau_j$. But $\pi^N_{h_j} : N^*_i \longrightarrow_{\Sigma^*} N_j$. Hence $N^*_i = \langle J^{E^N}_\nu, F\rangle$ where $F \neq \varnothing$. Similarly, $N_l = \langle J^{E^{N_l}}_{\nu_l}, F_l\rangle$, $(F_l = {E^{N_l}}_{\nu_l})$ for $i+1 \leq_{T^N} l \leq_{T^N} j$. If $i + 1 <_{T^N} k + 1 \leq j$ and k is active in I^N, we have $\kappa_k \geq \lambda_i > \tau$. Hence $\text{crit}(\pi^N_{i+1,j}) > \tau$. Thus $\tau = \tau_{i+1}$ and $\pi^N_{i+1,j} \upharpoonright \tau = \text{id}$. Clearly $[\kappa_i, \lambda_i) \cap \text{rng}(\pi^N_{h,i+1}) = \varnothing$. But $\kappa_i < \tau < \lambda_i$ and $\tau = \tau_{i+1} = \pi^N_{h,i+1}(\overline{\tau})$, where $\overline{\tau} = \tau_F$. Contradiction!

QED(Claim)

But then there is a truncation on the main branch of $I^N|j+1$, since otherwise:

$$N = N_1 = \langle J^{E^N}_{\nu_1}, F\rangle \text{ with } \tau_1 = \tau.$$

But τ is not a cardinal in N. Contradiction! Let $i+1$ be the final truncation point on the main branch of $I^N|j+1$. Let $h = T^N(i+1)$. Then, letting $\pi = \pi^*_{h,j}$, we have:

$$\pi : N^*_i \longrightarrow_{\Sigma^*} N_j,\ \pi \upharpoonright \tau = \text{id},\ \pi(p^*_{N^*_i}) = p^*_{N_i}.$$

Hence $\langle Q, F\rangle = \text{core}_\tau(N_j) = N^*_i$. But N^*_i si a proper segment of N_h. By our above arguments it again follows that N^*_i is a proper segment of N.

QED(7)

QED(Lemma 4.3.7)

Using the condensation lemma, we prove a sharper version of the initial segment condition for mice:

Lemma 4.3.8. *Let $N = \langle J^E_\nu, F\rangle$ be an active mouse. Let $\overline{\lambda} \in N$. Let $\overline{F} = F|\lambda$ be a full extender. Set:*

$$M = \langle J^E_{\overline{\nu}}, \overline{F}\rangle \textit{ where } \overline{\pi} : J^E_\tau \longrightarrow J^E \textit{ is the extension of } \vec{F}$$

. Then M is a a proper segment of N.

Proof. Let $\kappa = \text{crit}(F)$. Define $\tau = \tau_F, \lambda = \lambda_F, \nu = \nu_F$ as usual. Hence: $\tau = \kappa^{+N}, \lambda = F(\lambda)$. Then $\overline{\tau} = \tau_{\overline{F}}, \overline{\lambda} = \lambda_{\overline{F}}, \overline{\nu} = \nu_{\overline{F}}$. Let $\pi : J^E_\tau : J^E_\nu$ be the extension of F. Define: $\sigma : J^E_{\overline{\tau}} \longrightarrow J^E_\tau$ by:

$$\sigma(\overline{\pi}(f)(\alpha)) = \pi(f)(\alpha) \text{ for } \alpha < \overline{\lambda}, f \in J^E_\tau, \text{dom}(f) = u.$$

Then $\overline{\lambda} = \text{crit}(\lambda), \sigma(\overline{\lambda})$ and σ is Σ_0-preserving, where:

$$\rho^\omega_M \le \overline{\lambda} \text{ and } \varnothing \notin R^{(\overline{\lambda})}_M.$$

This is because $\overline{\pi}$ is $\Sigma_1(M)$ and each element of M has the form $\overline{\pi}(f)(\alpha)$ where $f \in J^E_\tau$ and $\alpha < \overline{\lambda}$. It follows easily that σ witnesses the phalanx $\langle N, M, \overline{\lambda}\rangle$. Applying the condensation lemma, we see that one of the possibilities (a), (b), (c) holds. (c) cannot hold since $\overline{\lambda}$ is a limit cardinal in M. (a) cannot hold, since $M \in N$ by the initial segment condition. If (a) holds, we would have: $\sigma(p^*_M) = p^*_N, \sigma \upharpoonright \overline{\lambda} = \text{id}$, where σ is Σ^*-preserving. But then $\rho^\omega_M = \rho^\omega_N$. Let $\rho = \rho^\omega_N$. Let A be $\Sigma^*(N)$ in p^*_N such that $A \cap \rho \notin N$. Let $\overline{A}$ be $\Sigma^*(M)$ in p^*_M by the same defition. Then:

$$A \cap \rho = \overline{A} \cap \rho \in \underline{\Sigma}^*(M) \subset N.$$

Contradiction! Thus, only the possibility (b) remains.

QED(Lemma 4.3.8)

As a corollary of the proof of Lemma 4.3.7, we get:

Lemma 4.3.9. *Let $\lambda = \rho^n_N < \mathrm{On} \cap M$ $(n \leq \omega)$, where N is critic at λ. Let $M = \mathrm{core}_\lambda(N)$. Let $\mu =: \lambda^{+N}$ (with $\mu =: \mathrm{On}_N$ if N has a largest cardinal). Then $\mu = \lambda^{+M}$ and $N||\mu = M||\mu$.*

Proof. Let $\sigma = \sigma^\lambda_N$. Then σ witnesses the phalanx $\langle N, M, \lambda\rangle$. Then (b) and (c) in Lemma 4.3.7 cannot hold, since otherwise $a \in N$ where $a \subset \lambda$ is $\underset{\sim}{\Sigma}^{(m)}_1(M)$ such that $a \notin N$. Contradiction! Coiterate $\langle N, M, \lambda\rangle, N$ to get I^M, I^N as in the proof of Lemma 4.3.7. Then Cases 2 and 3 cannot hold, since otehrwise (b) or (c) would hold. Hence Case 1 holds -i.e. $M_\xi = N_\xi$ and I^N has not truncation on the main branch. We know that $\kappa_i \geq \lambda$ on the main branch of I^M. Hence $\lambda = \rho^n_{N_\eta}$. But $\rho^n_{M_\eta} = \rho^n_{N_\eta}$. Hence $\lambda = \rho^n_{N_\eta}$. But then $\kappa_i \geq \lambda$ on the main branch of I^N, since otherwise $\lambda < \pi^N_{0,\eta}(\lambda) = \rho^n_{N_\eta}$. Since there is no truncation on the main branch, we have $\lambda_i \geq \mu$. Hence $M||\mu = M_\eta||\mu = N_\xi||\mu = N||\mu$, where $\mu = \lambda^{+M_\eta}$.

QED(Lemma 4.3.9)