# Background material 2

## Contact manifolds, fillings, cobordisms

A Liouville vector field $V$ on $(W^{2n}, \omega)$ satisfies

$$\mathcal{L}_V \omega = \omega$$

# Background material 2

## Contact manifolds, fillings, cobordisms

A Liouville vector field $V$ on $(W^{2n}, \omega)$ satisfies

$$\mathcal{L}_V \omega = \omega$$

$\Leftrightarrow$ the 1-form $\lambda := \iota_V \omega$ satisfies

$$d\lambda = \omega$$

We call $\lambda$ a Liouville form.

# Background material 2

## Contact manifolds, fillings, cobordisms

A Liouville vector field $V$ on $(W^{2n}, \omega)$ satisfies

$$\mathcal{L}_V \omega = \omega$$

$\Leftrightarrow$ the 1-form $\lambda := \iota_V \omega$ satisfies

$$d\lambda = \omega$$

We call $\lambda$ a Liouville form.

**Definition**
A component $M \subset \partial(W, \omega)$ is convex/concave if near $M$ there is a Liouville vector field $V$ pointing transversely outward/inward.

# Background material 2

## Contact manifolds, fillings, cobordisms

A Liouville vector field $V$ on $(W^{2n}, \omega)$ satisfies

$$\mathcal{L}_V \omega = \omega$$

$\Leftrightarrow$ the 1-form $\lambda := \iota_V \omega$ satisfies

$$d\lambda = \omega$$

We call $\lambda$ a Liouville form.

**Definition**
A component $M \subset \partial(W, \omega)$ is convex/concave if near $M$ there is a Liouville vector field $V$ pointing transversely outward/inward.

Equivalently, $\lambda := \iota_V \omega$ satisfies

$$\boxed{\lambda \wedge (d\lambda)^{n-1} > 0}$$

on $\pm M$.

# Background material 2

## Contact manifolds, fillings, cobordisms

A Liouville vector field $V$ on $(W^{2n}, \omega)$ satisfies

$$\mathcal{L}_V \omega = \omega$$

$\Leftrightarrow$ the 1-form $\lambda := \iota_V \omega$ satisfies

$$d\lambda = \omega$$

We call $\lambda$ a Liouville form.

**Definition**
A component $M \subset \partial(W, \omega)$ is convex/concave if near $M$ there is a Liouville vector field $V$ pointing transversely outward/inward.

Equivalently, $\lambda := \iota_V \omega$ satisfies

$$\boxed{\lambda \wedge (d\lambda)^{n-1} > 0}$$

on $\pm M$. This means $\lambda|_{TM}$ is a (positive/negative) **contact form**, with **contact structure**

$$\xi = \ker \lambda \subset TM$$

# Background material 2

## Contact manifolds, fillings, cobordisms

A Liouville vector field $V$ on $(W^{2n}, \omega)$ satisfies

$$\mathcal{L}_V \omega = \omega$$

$\Leftrightarrow$ the 1-form $\lambda := \iota_V \omega$ satisfies

$$d\lambda = \omega$$

We call $\lambda$ a Liouville form.

**Definition**
A component $M \subset \partial(W, \omega)$ is convex/concave if near $M$ there is a Liouville vector field $V$ pointing transversely outward/inward.

Equivalently, $\lambda := \iota_V \omega$ satisfies

$$\boxed{\lambda \wedge (d\lambda)^{n-1} > 0}$$

on $\pm M$. This means $\lambda|_{TM}$ is a (positive/negative) **contact form**, with **contact structure**

$$\xi = \ker \lambda \subset TM$$

**Fact**: $\omega$ determines $\xi$ uniquely up to isotopy.

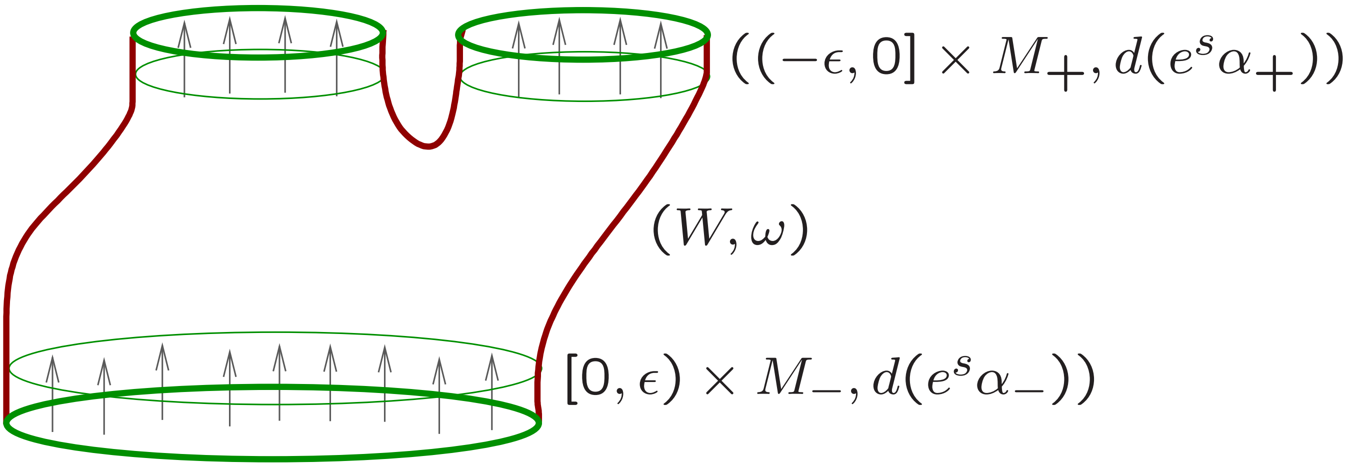
**Definition**

A symplectic cobordism from

$(M_-, \xi_- = \ker \alpha_-)$ to $(M_+, \xi_+ = \ker \alpha_+)$:

$$\text{“}\partial(W, \omega) = (-M_-, \xi_-) \sqcup (M_+, \xi_+)\text{”}$$

- Convex at $M_+$: $\omega = d\lambda$ with $\lambda|_{TM_+} = \alpha_+$
- Concave at $M_-$: $\omega = d\lambda$ with $\lambda|_{TM_-} = \alpha_-$

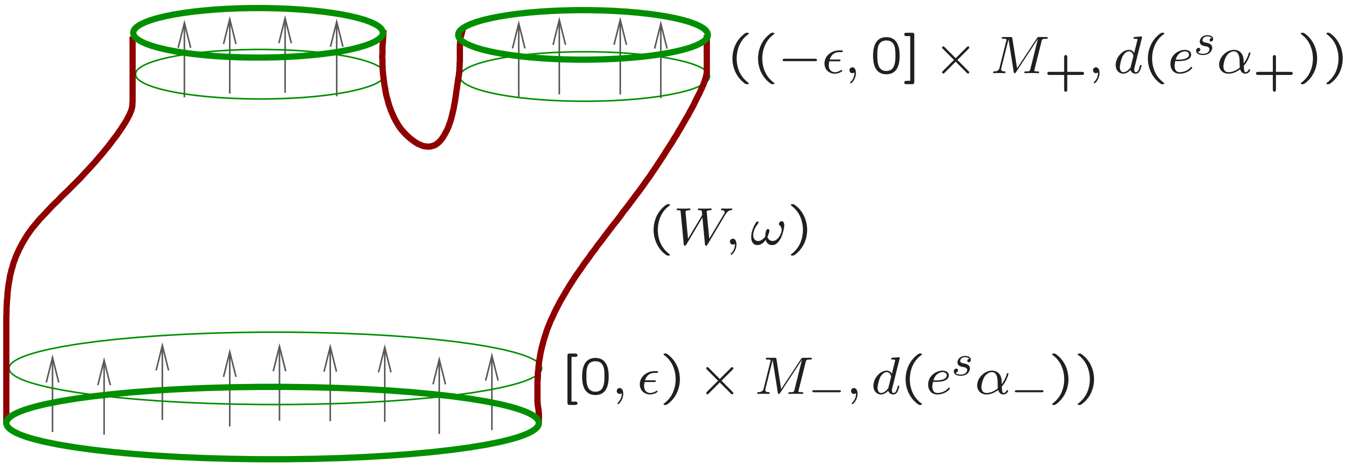


**Definition**

A symplectic cobordism from $(M_-, \xi_- = \ker \alpha_-)$ to $(M_+, \xi_+ = \ker \alpha_+)$:

$$\text{“}\partial(W, \omega) = (-M_-, \xi_-) \sqcup (M_+, \xi_+)\text{”}$$

- Convex at $M_+$: $\omega = d\lambda$ with $\lambda|_{TM_+} = \alpha_+$
- Concave at $M_-$: $\omega = d\lambda$ with $\lambda|_{TM_-} = \alpha_-$



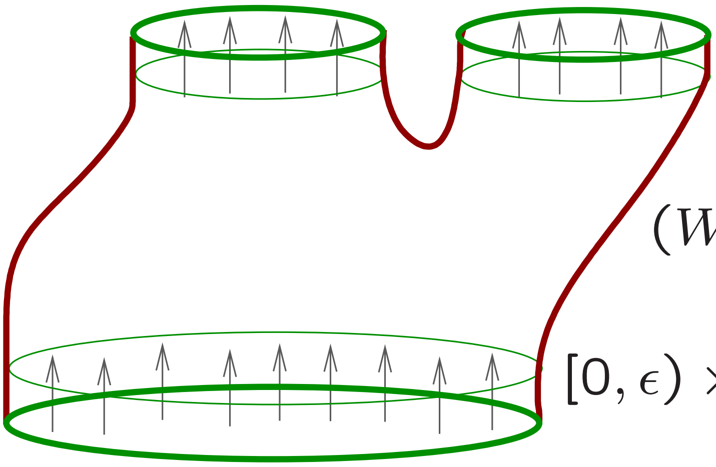

Case $M_- = \emptyset$:
$(W, \omega)$ is a symplectic filling of $(M_+, \xi_+)$

**Definition**

A symplectic cobordism from $(M_-, \xi_- = \ker \alpha_-)$ to $(M_+, \xi_+ = \ker \alpha_+)$:

$$\text{“}\partial(W,\omega) = (-M_-, \xi_-) \sqcup (M_+, \xi_+)\text{”}$$

- Convex at $M_+$: $\omega = d\lambda$ with $\lambda|_{TM_+} = \alpha_+$
- Concave at $M_-$: $\omega = d\lambda$ with $\lambda|_{TM_-} = \alpha_-$




Case $M_- = \emptyset$:
$(W,\omega)$ is a symplectic filling of $(M_+, \xi_+)$

Case $M_+ = \emptyset$:
$(W,\omega)$ is a symplectic cap for $(M_-, \xi_-)$

## Some results on contact 3-manifolds $(M, \xi)$

1. Every $M$ admits a contact structure $\xi$.
   *(Martinet '71)*

## Some results on contact 3-manifolds $(M, \xi)$

1. Every $M$ admits a contact structure $\xi$.

   *(Martinet '71)*

2. Any $\xi$ on any $M$ is homotopic to a non-fillable ("overtwisted") $\xi'$.

   *(Gromov '85 + Eliashberg '90)*

## Some results on contact 3-manifolds $(M, \xi)$

1. Every $M$ admits a contact structure $\xi$.

   *(Martinet '71)*

2. Any $\xi$ on any $M$ is homotopic to a non-fillable ("overtwisted") $\xi'$.

   *(Gromov '85 + Eliashberg '90)*

3. Some $M$ admit no fillable $\xi$ *(Lisca '98)*, and some admit only overtwisted $\xi$.

   *(Etnyre-Honda '01)*

## Some results on contact 3-manifolds $(M, \xi)$

1. Every $M$ admits a contact structure $\xi$.

   *(Martinet '71)*

2. Any $\xi$ on any $M$ is homotopic to a non-fillable ("overtwisted") $\xi'$.

   *(Gromov '85 + Eliashberg '90)*

3. Some $M$ admit no fillable $\xi$ *(Lisca '98)*, and some admit only overtwisted $\xi$.

   *(Etnyre-Honda '01)*

4. Every $(M, \xi)$ admits many symplectic caps.

   *(Etnyre-Honda '02)*

## Some results on contact 3-manifolds $(M, \xi)$

1. Every $M$ admits a contact structure $\xi$.

   *(Martinet '71)*

2. Any $\xi$ on any $M$ is homotopic to a non-fillable ("overtwisted") $\xi'$.

   *(Gromov '85 + Eliashberg '90)*

3. Some $M$ admit no fillable $\xi$ *(Lisca '98)*, and some admit only overtwisted $\xi$.

   *(Etnyre-Honda '01)*

4. Every $(M, \xi)$ admits many symplectic caps.

   *(Etnyre-Honda '02)*

5. Every overtwisted $(M, \xi)$ admits a symplectic cobordism to every other $(M', \xi')$.

   *(Etnyre-Honda '02)*

## Some results on contact 3-manifolds $(M, \xi)$

1. Every $M$ admits a contact structure $\xi$.

   *(Martinet '71)*

2. Any $\xi$ on any $M$ is homotopic to a non-fillable ("overtwisted") $\xi'$.

   *(Gromov '85 + Eliashberg '90)*

3. Some $M$ admit no fillable $\xi$ *(Lisca '98)*, and some admit only overtwisted $\xi$.

   *(Etnyre-Honda '01)*

4. Every $(M, \xi)$ admits many symplectic caps.

   *(Etnyre-Honda '02)*

5. Every overtwisted $(M, \xi)$ admits a symplectic cobordism to every other $(M', \xi')$.

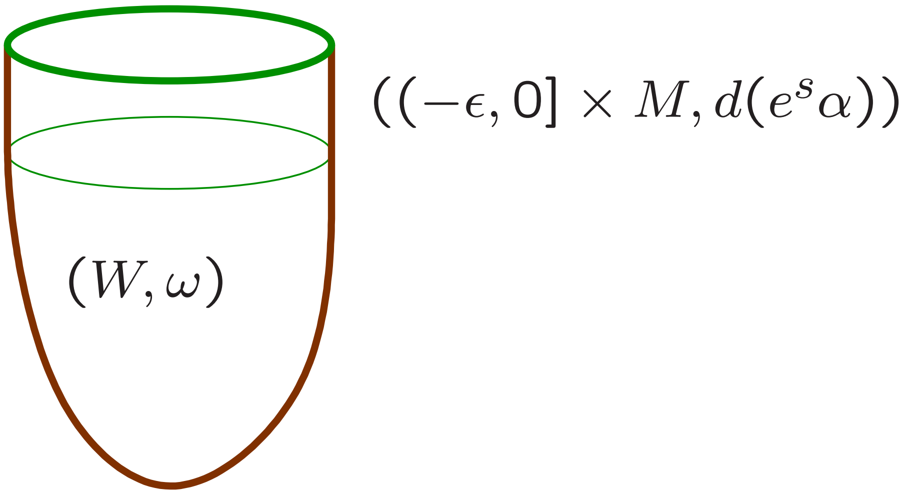
   *(Etnyre-Honda '02)*

6. All symplectic fillings of $(S^3, \xi_{\text{std}})$ are $(B^4, \omega_{\text{std}})$, up to symplectic deformation equivalence and blowup.

   *(Gromov '85)*

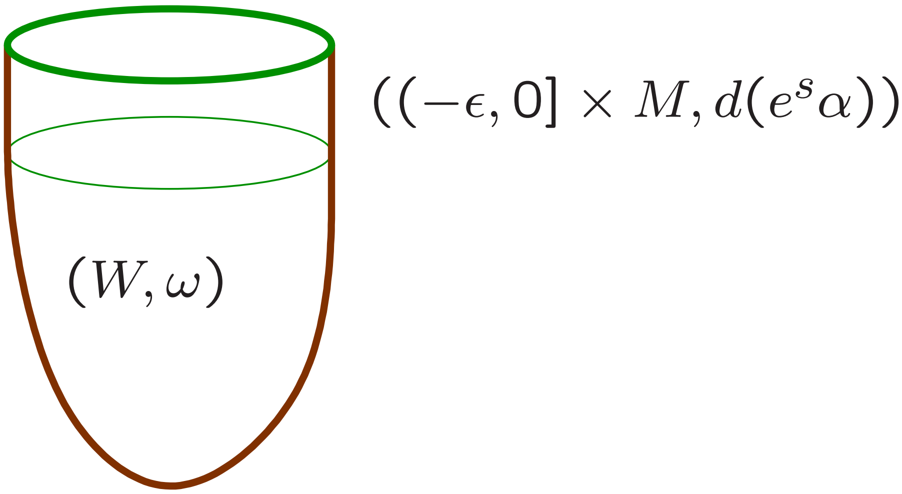
**Remark**

Topologically, "$\partial X \cong S^3$" imposes no restrictions on $X$. Symplectic topology is much more rigid.



**Remark**

Topologically, "$\partial X \cong S^3$" imposes no restrictions on $X$. Symplectic topology is much more rigid.




In Lecture 5, we will prove:

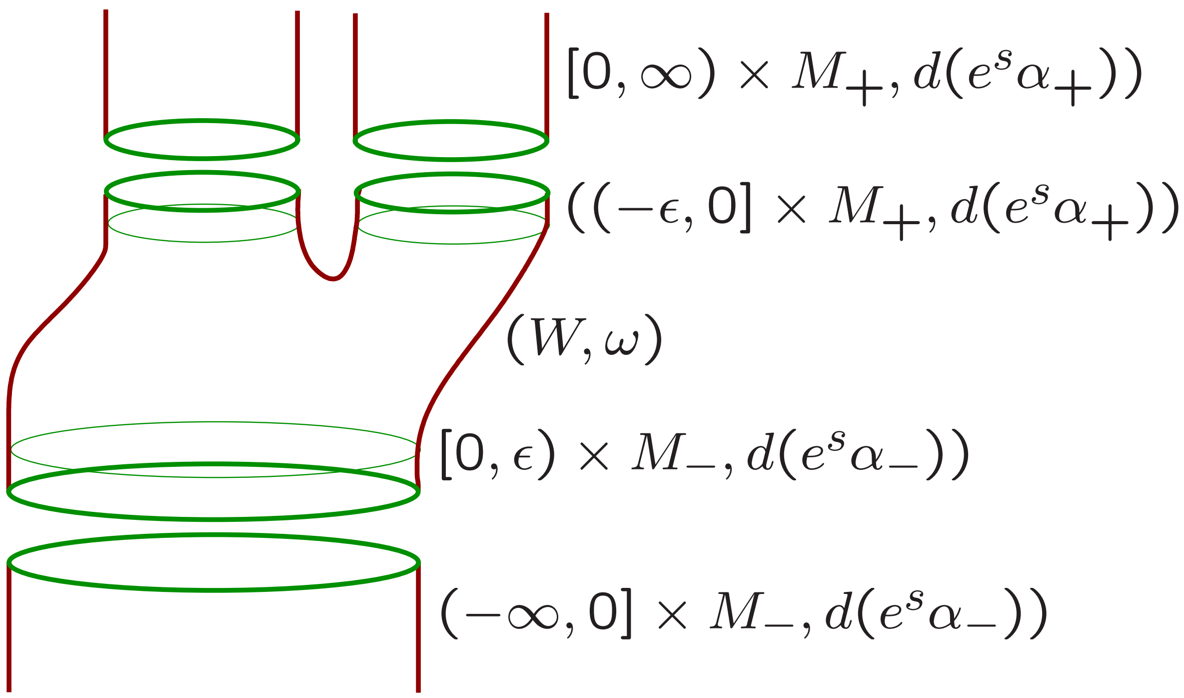
**Theorem**

*Symplectic fillings of $(S^3, \xi_{\text{std}})$, $(S^1 \times S^2, \xi_{\text{std}})$ and $(L(k, k-1), \xi_{\text{std}})$ are unique up to symplectic deformation and blowup.*

*(Gromov '85, Eliashberg '90, Lisca '08, W. '10)*

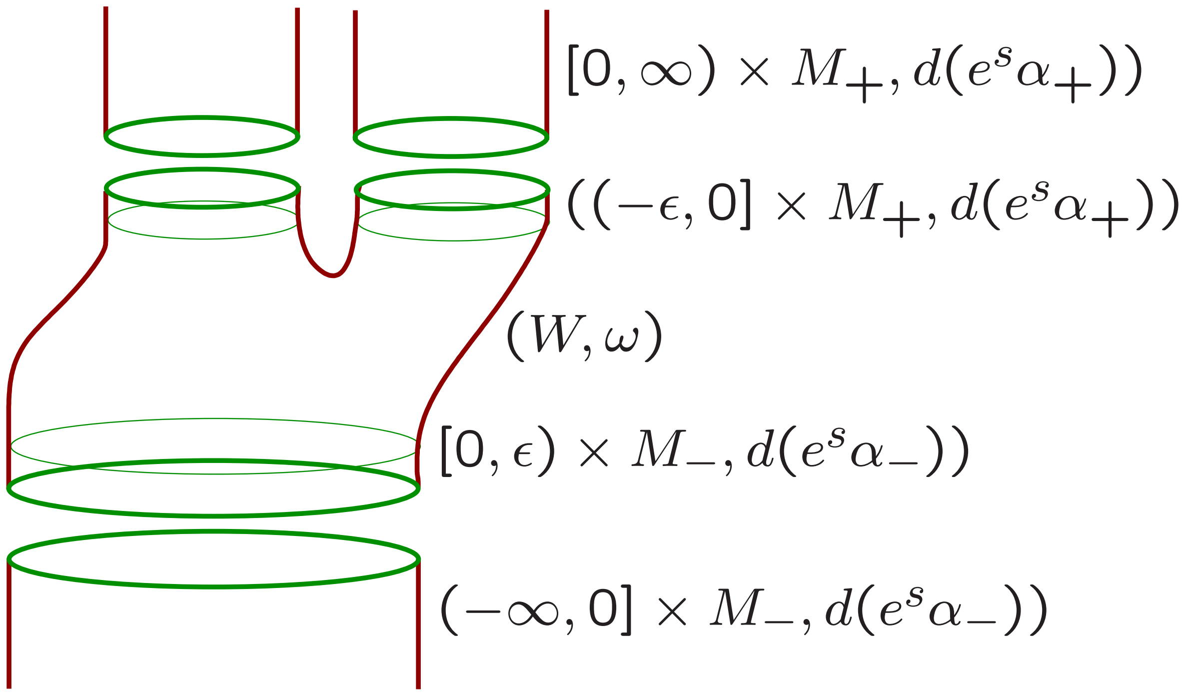
## Asymptotically cylindrical holomorphic curves

$(W, \omega) \rightsquigarrow$ **completion** $(\widehat{W}, \hat{\omega})$



# Asymptotically cylindrical holomorphic curves

$(W, \omega) \rightsquigarrow$ **completion** $(\widehat{W}, \hat{\omega})$

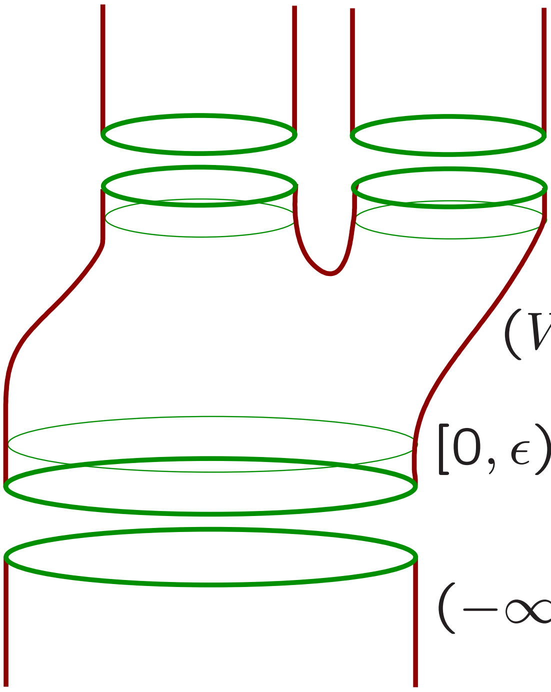



Trivial case: **symplectisation** of $(M, \xi = \ker \alpha)$:

$$(\mathbb{R} \times M, d(e^s \alpha))$$

## Asymptotically cylindrical holomorphic curves

$(W,\omega) \rightsquigarrow$ **completion** $(\widehat{W},\hat{\omega})$




Trivial case: **symplectisation** of $(M,\xi = \ker\alpha)$:

$$(\mathbb{R}\times M, d(e^s\alpha))$$

Let $\mathcal{J}(\alpha) :=$ $\mathbb{R}$-invariant a.c.s.'s $J$ with:

- $J(\partial_s) = R_\alpha$, the **Reeb vector field on** $M$:
$$d\alpha(R_\alpha,\cdot)\equiv 0, \quad \alpha(R_\alpha)\equiv 1$$
- $J|_\xi$ is compatible with $d\alpha|_\xi$

Given Reeb orbit $\gamma : S^1 \to M$ of period $T > 0$,

$$\mathbb{R} \times S^1 \to \mathbb{R} \times M : (s,t) \mapsto (Ts, \gamma(t))$$

is a $J$-holomorphic "orbit cylinder".

Given Reeb orbit $\gamma : S^1 \to M$ of period $T > 0$,

$$\mathbb{R} \times S^1 \to \mathbb{R} \times M : (s,t) \mapsto (Ts, \gamma(t))$$

is a $J$-holomorphic "orbit cylinder".

Choose $J$ on $\widehat{W}$ such that $\omega$-compatible and $J \in \mathcal{J}(\alpha_\pm)$ on ends. We consider punctured, *asymptotically cylindrical* $J$-holomorphic curves

$$u : \dot{\Sigma} = \Sigma \setminus \Gamma \to \widehat{W}$$

approaching Reeb orbits in $\{\pm\infty\} \times M_\pm$ at the punctures.

Given Reeb orbit $\gamma : S^1 \to M$ of period $T > 0$,

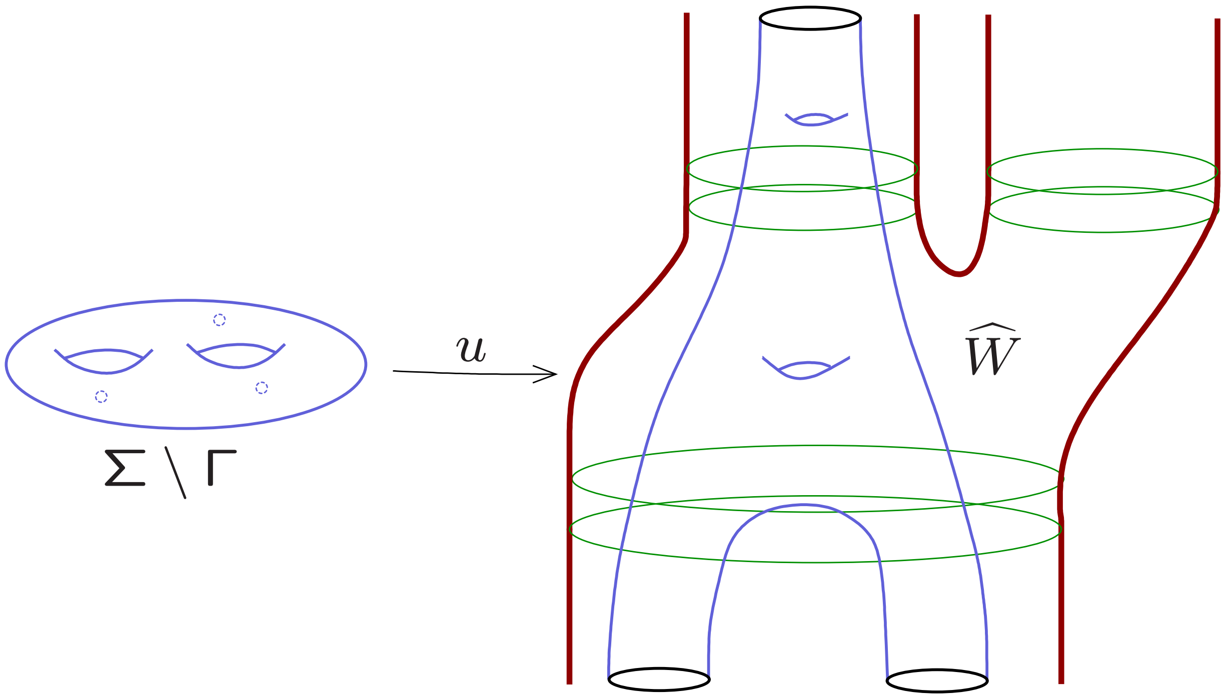
$$\mathbb{R} \times S^1 \to \mathbb{R} \times M : (s,t) \mapsto (Ts, \gamma(t))$$

is a $J$-holomorphic "orbit cylinder".

Choose $J$ on $\widehat{W}$ such that $\omega$-compatible and $J \in \mathcal{J}(\alpha_\pm)$ on ends. We consider punctured, *asymptotically cylindrical* $J$-holomorphic curves

$$u : \dot{\Sigma} = \Sigma \setminus \Gamma \to \widehat{W}$$

approaching Reeb orbits in $\{\pm\infty\} \times M_\pm$ at the punctures.



## Virtual dimension

Fix a choice of trivialisation $\tau$ of $\gamma^*\xi_\pm \to S^1$ for every Reeb orbit $\gamma$.

## Virtual dimension

Fix a choice of trivialisation $\tau$ of $\gamma^*\xi_\pm \to S^1$ for every Reeb orbit $\gamma$.

Near a simple curve $u : \dot{\Sigma} \to \widehat{W}$ asymptotic to nondegenerate Reeb orbits $\{\gamma_z\}_{z\in\Gamma^\pm}$, the moduli space (for generic $J$) has dimension

$$\text{ind}(u) := (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(u^*T\widehat{W}) + \sum_{z\in\Gamma^+} \mu_{\text{CZ}}^\tau(\gamma_z) - \sum_{z\in\Gamma^-} \mu_{\text{CZ}}^\tau(\gamma_z),$$

where

- $c_1^\tau(u^*T\widehat{W})$ is the relative first Chern number of $(u^*T\widehat{W}, J) \to \dot{\Sigma}$
- $\mu_{\text{CZ}}^\tau(\gamma)$ is the Conley-Zehnder index of $\gamma$

## Virtual dimension

Fix a choice of trivialisation $\tau$ of $\gamma^*\xi_\pm \to S^1$ for every Reeb orbit $\gamma$.

Near a simple curve $u : \dot{\Sigma} \to \widehat{W}$ asymptotic to nondegenerate Reeb orbits $\{\gamma_z\}_{z\in\Gamma^\pm}$, the moduli space (for generic $J$) has dimension

$$\text{ind}(u) := (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(u^*T\widehat{W}) + \sum_{z\in\Gamma^+}\mu_{\text{CZ}}^\tau(\gamma_z) - \sum_{z\in\Gamma^-}\mu_{\text{CZ}}^\tau(\gamma_z),$$

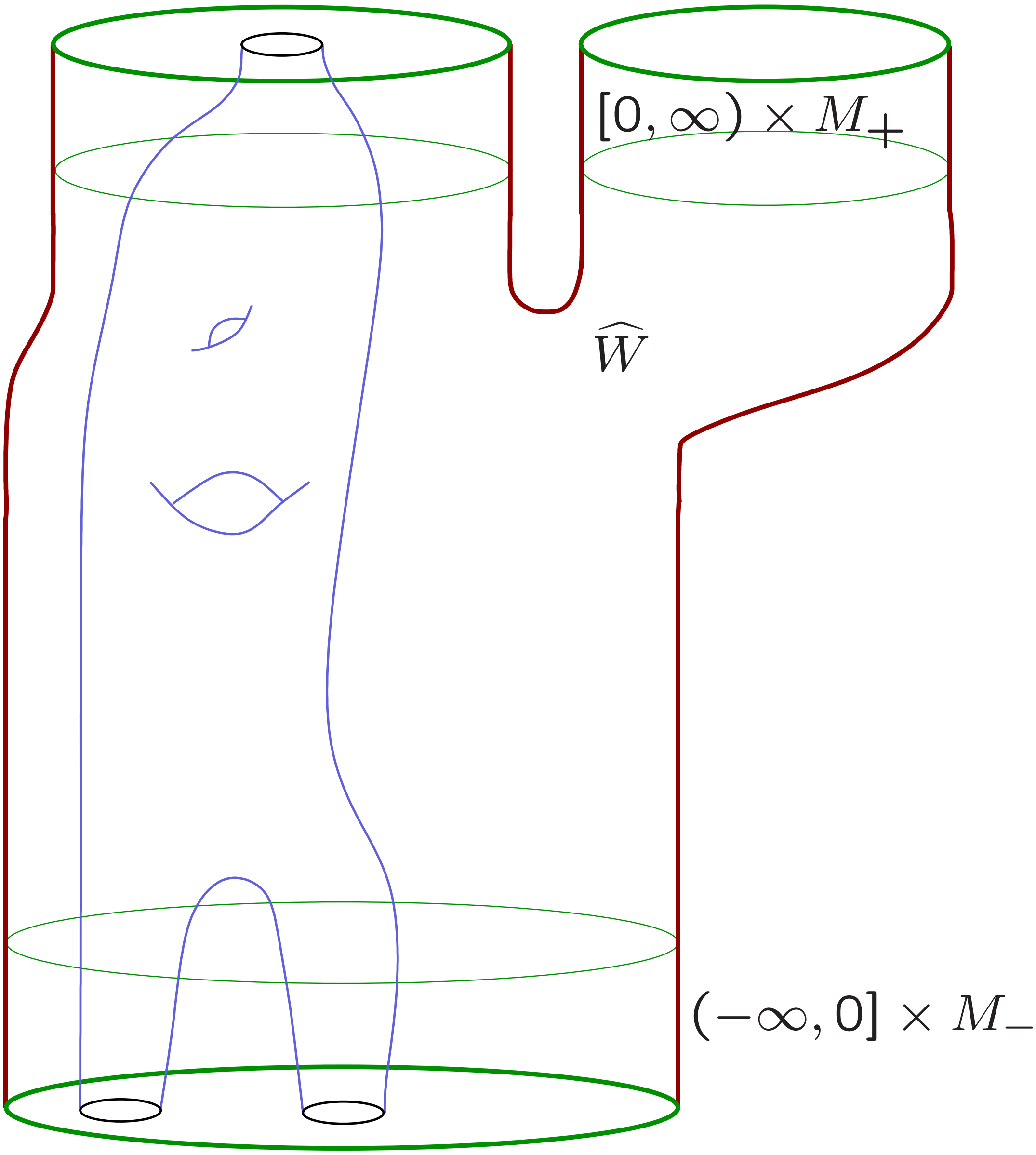
where

- $c_1^\tau(u^*T\widehat{W})$ is the relative first Chern number of $(u^*T\widehat{W}, J) \to \dot{\Sigma}$
- $\mu_{\text{CZ}}^\tau(\gamma)$ is the Conley-Zehnder index of $\gamma$

The sum is independent of $\tau$.

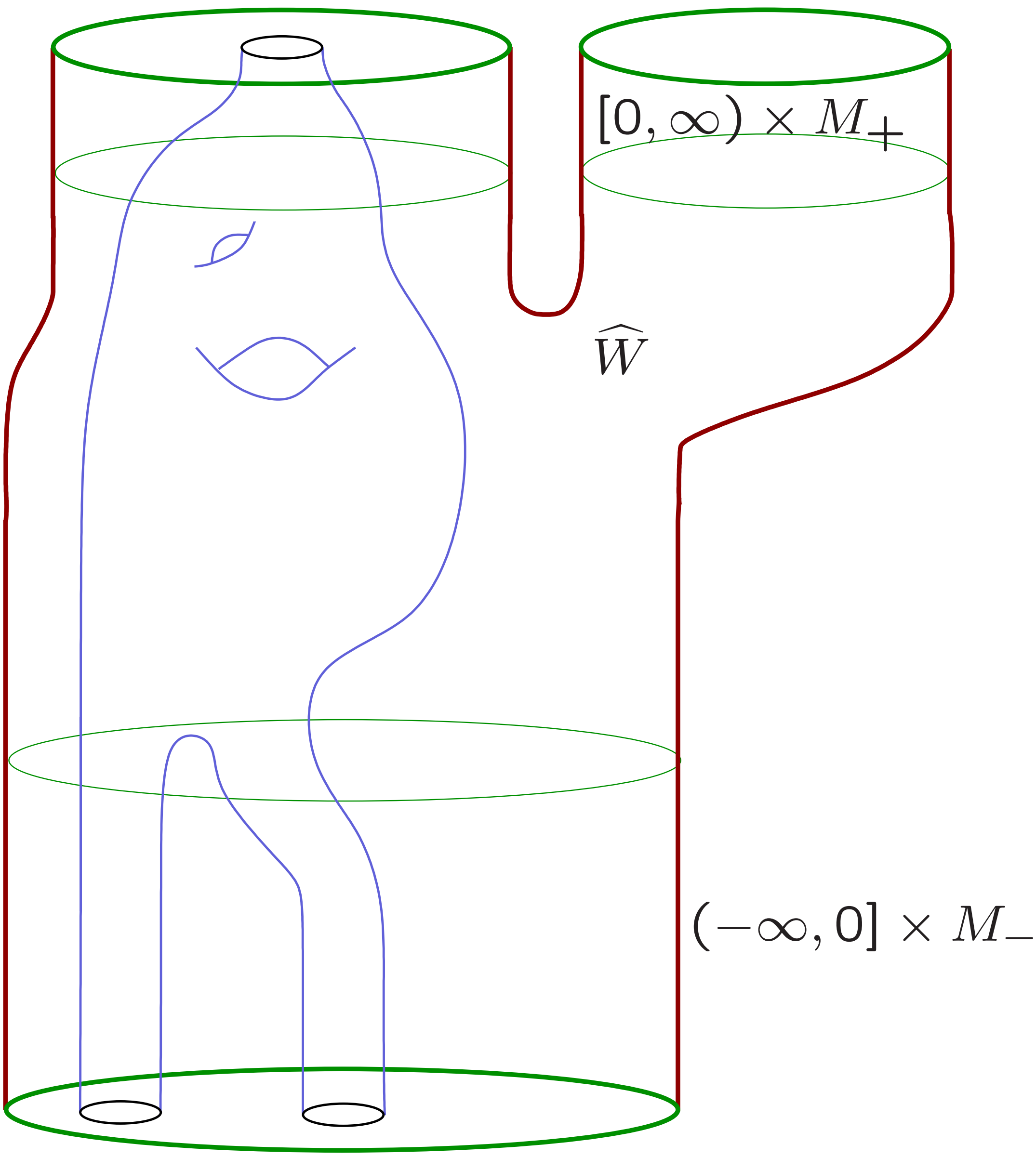
## Compactification

Sequences can converge to (nodal) **$J$-holomorphic buildings**:



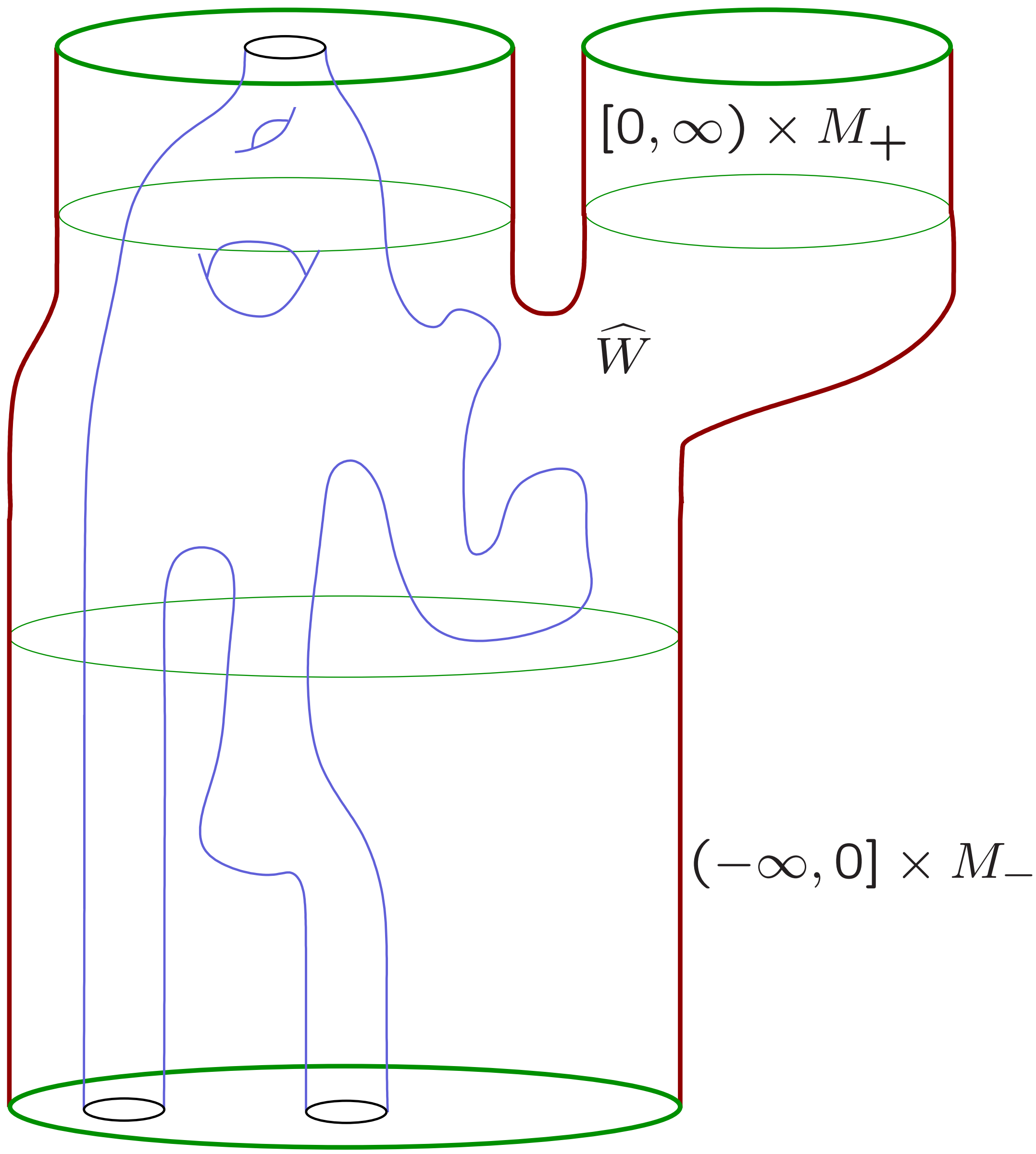
## Compactification

Sequences can converge to (nodal) **$J$-holomorphic buildings**:



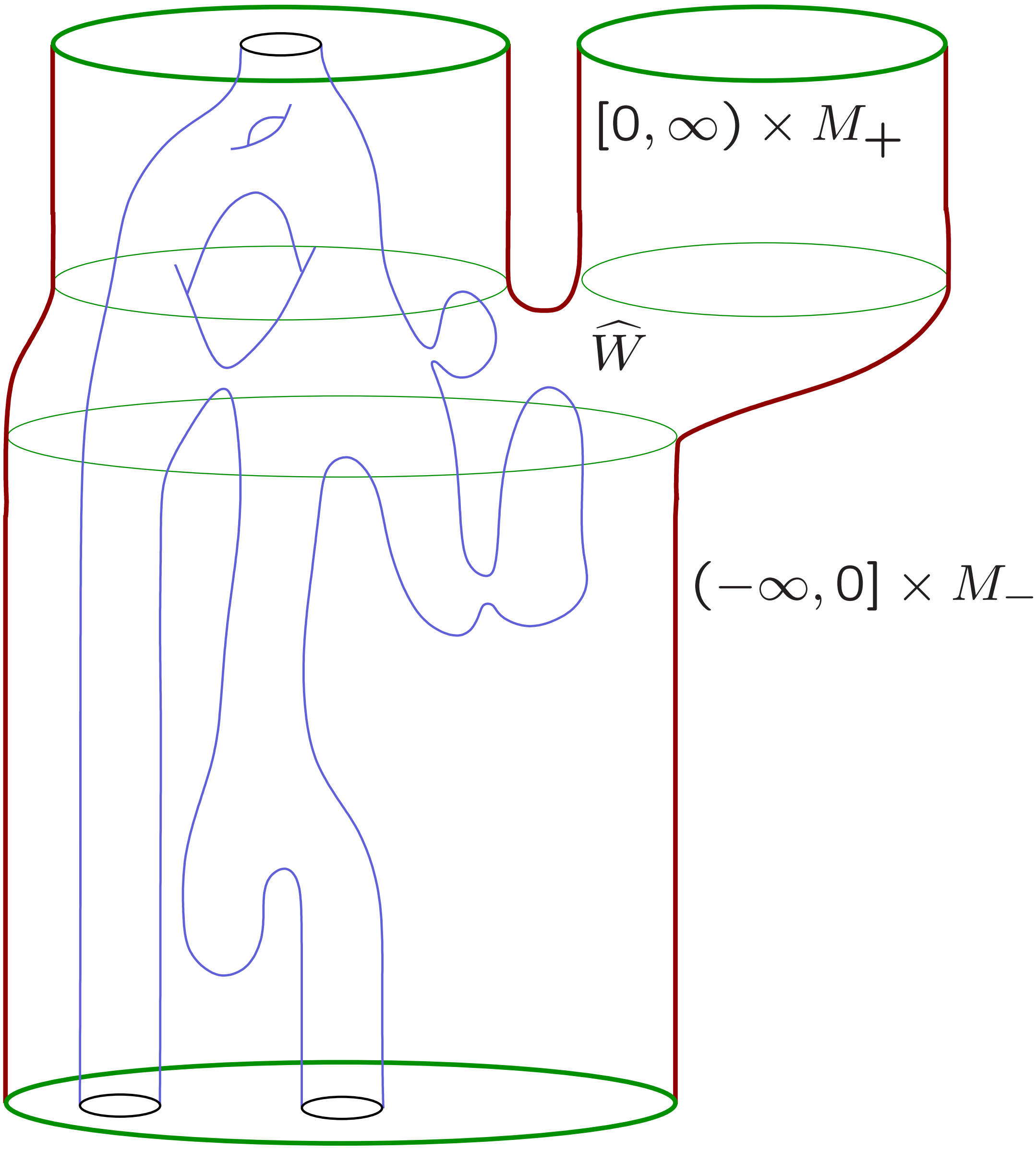
## Compactification

Sequences can converge to (nodal)
***J*-holomorphic buildings**:



## Compactification

Sequences can converge to (nodal) **$J$-holomorphic buildings**:



## Compactification

Sequences can converge to (nodal) **$J$-holomorphic buildings**: