Liouville and Weinstein manifolds

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Contents

Disclaimer: The following lecture notes are very rough. Some parts are rather precise while others are very sketchy, probably because our understanding is also uneven. The attribution of results might not always be the most appropriate. We apologize for all that and hope that these notes may still be useful for some people.

Notations:

- $\mathrm{Op}(A)$ denotes an unspecified open neighborhood of A.
- pr_i denotes the projection on the i-th factor of a product.

1 Liouville and Weinstein manifolds

1.1 Symplectic convexity

A *symplectic form* on a manifold W is a closed non-degenerate 2-form. If it exists then W has even dimension $2n$ and admits an almost complex structure. A vector field X is called *Liouville* if $X.\omega = \omega$ (the dot stands for Lie derivative). From the Lie-Cartan formula, we have $\mathrm{d}\lambda = \omega$ where $\lambda = X \lrcorner \omega$ is the associated *Liouville form* ($\lrcorner$ stands for interior product). The flow ϕ_X^t of X exponentially expands both λ and ω: $(\phi_X^t)^*\lambda = e^t\lambda$, $(\phi_X^t)^*\omega = e^t\omega$. Liouville vector fields have a *transverse contact structure*:

- if $i : \Sigma \to W$ is a hypersurface transverse to X, then $\alpha = i^*\lambda$ is a contact form on Σ:
$$\alpha \wedge (\mathrm{d}\alpha)^{n-1} = i^*(\lambda \wedge (\mathrm{d}\lambda)^{n-1}) = \frac{1}{n} i^*(X \lrcorner \omega^n).$$
The hyperplane field $\xi = \ker \alpha$ is called a *contact structure*.
- if $h : \Sigma \to \Sigma'$ is the holonomy diffeomorphism between two hypersurfaces transverse to X then $h_*\xi = \xi'$ where ξ and ξ' are the contact structures induced respectively on Σ and Σ' as above: there exists a smooth function $t : \Sigma \to \mathbb{R}$ so that, $\forall p \in \Sigma$, $h(p) = \phi_X^{t(p)}(p)$, then we have
$$h^*\lambda = e^{t(p)}\lambda + \lambda(X)\mathrm{d}t = e^{t(p)}\lambda.$$

Note that the contact form is not preserved by holonomy, only its kernel is.

A cooriented hypersuface Σ in a symplectic manifold (W, ω) is called *ω-convex* if there exists a Liouville vector field transverse to Σ (one may require X to be defined only near Σ, or on the whole W, leading to different notions). The first systematic study of symplectic convexity appeared in [EG91], but this notion had been considered previously especially by Weinstein. The Reeb vector field R_α of $\alpha = i^*(X \lrcorner \omega)$ (defined by $R_\alpha \lrcorner \mathrm{d}\alpha = 0$ and $\alpha(R_\alpha) = 1$) spans the *characteristic foliation* $\ker(i^*\omega)$, and this implies particular dynamical properties of this foliation. Weinstein conjectured in [Wei79] that any closed ω-convex hypersurface has at least one closed orbit. Viterbo proved it in the case of $\mathbb{R}^{2n}$ (see [Vit87]), while Ginzburg and Herman found examples of hypersurfaces in $\mathbb{R}^{2n}$ without closed characteristics, hence not ω-convex. There are much simpler obstructions to ω-convexity. First observe that the characteristic foliation of a cooriented hypersurface is naturally oriented by the rule: a vector v spanning $\ker(i^*\omega)$ is positive if $\omega(n, v) > 0$ for n positively transverse to Σ. Second, for such a positive v, and λ a convex, resp. concave Liouville form, we have $\omega(X, v) = \lambda(v) > 0$, resp. < 0. Third, if γ is a homologically trivial periodic orbit, then the action $\int_\gamma \lambda$ does not depend on the choice of a primitive λ for ω. If this action is zero or if there exists two such orbits with actions of opposite signs, then Σ is neither convex nor concave. This kind of consideration have been pushed much further by McDuff, who gave a complete characterization of (local) ω-convexity for closed hypersurfaces in terms of dynamical properties of the characteristic foliation (see [McD87, section 5]). In this spirit, see also exercise 1 below. The first systematic study of symplectic convexity appeared in [EG91].

Examples 1) In $\mathbb{R}^{2n}$ we have the symplectic form $\omega_0 = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$ and the radial Liouville vector field $X = \frac{1}{2}(\sum x_i \partial_{x_i} + y_i \partial_{y_i})$. A closed hypersurface transverse to X is diffeomorphic to a sphere and bounds a domain in $\mathbb{R}^{2n}$ which is star-shaped from the origin. The contact structure induced is the standard contact structure on S^{2n-1}. On the unit sphere, the Reeb flow spans the Hopf fibration: $R_\alpha = 2(x_i \partial_{y_i} - y_i \partial_{x_i}) = 2iz_i$ and $\phi_t^{R_\alpha}(z_i) = e^{2it}(z_i)$ is totally periodic of period π. One can get only n non-degenerate orbits by considering a well-chosen ellipsoïd instead. It was proved by Rabinowitz in [Rab79] that any such hypersurface admits at least one closed characteristic.

2) The cotangent bundle $\mathrm{T}^* M$ has a canonical Liouville form $\lambda = p\mathrm{d}q$. The corresponding Liouville vector field is radial in each fiber $X = p\partial_p$. Closed hypersurfaces transverse to X are boundaries of tubular neighborhoods of the zero section. If the intersection with each fiber is an ellipsoïd, then this corresponds to the unit disc bundle for some riemannian metric on M and the Reeb flow is the geodesic flow of this metric: in local coordinates $g = g_{ij}$, $\omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$, $\Sigma = \{\sum g_{ij} p_i p_j = 1\}$, and the Reeb vector field writes

$$R_\alpha = \sum_{i,j} \left(g_{ij} p_i \partial_{q_j} - \frac{1}{2} \sum_k \frac{\partial g_{ij}}{\partial q_k} p_i p_j \partial_{p_k} \right).$$

After some computation (exercise 2) we can get from this the second order equation for geodesics:

$$\ddot{q}_i + \sum_{jk} \Gamma^i_{jk} \dot{q}_j \dot{q}_k = 0,$$

where

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lj}}{\partial q_k} + \frac{\partial g_{lk}}{\partial q_j} - \frac{\partial g_{jk}}{\partial q_l} \right).$$

3) The symplectization $\mathrm{S}_\xi M = \{\beta \in \mathrm{T}^* M, \ker \beta = \xi\}$ of a contact manifold (M, ξ) is endowed with λ and X restricted from $\mathrm{T}^* M$. It is the local model near any ω-convex hypersurface. The sections of the bundle $\mathrm{S}_\xi M \to M$ are precisely the contact forms for ξ. The choice of a contact form α provides a trivialization $\mathrm{S}_\xi M = \mathbb{R}_+ \times M$ where $\lambda = r\alpha$ with the coordinate in $\mathbb{R}_+$ denoted by r. The functions $H : \mathrm{S}_\xi M \to \mathbb{R}$ satisfying $X.H = H$ are called homogeneous or linear, they read $H(r, p) = rh(p)$ for some function $h : M \to \mathbb{R}$ in a trivialization. The Hamiltonian vector field of H lifts a contact vector field (see exercise 3, which in a trivialization is the one associated to h and α. More generally every contact geometric property can be rephrased in terms of equivariant symplectic geometry of the symplectization. $\mathbb{R}^{2n} \setminus 0$ is the symplectization of S^{2n-1} and $\mathrm{T}^* M \setminus M$ (zero section removed) is the symplectization of the cosphere bundle $\mathrm{S}\,\mathrm{T}^* M$ (i.e., the space of cooriented hyperplanes tangent to M).

4) From [Wei79, section 3]. Start with the round sphere S^3 in $\mathbb{C}^2$. In an arbitrarily small neighborhood of a Hopf fiber, twist the sphere so that it coincides with the round sphere in a neighborhood of this hopf fiber but with opposite coorientation. Then this hypersurface is neither convex nor concave because it has closed characteristics of both positive and negative action (see the discussion above).

1.2 Liouville structures

Definition 1.1. *A* Liouville structure *on a compact cobordism is a symplectic form ω together with a Liouville vector field X which is positively transverse to the boundary (i.e., inward pointing along $\partial_- W$ and outward pointing along $\partial_+ W$).*

By Stokes' formula, such a Liouville structure may exist only if $\partial_+ W \neq \emptyset$:

$$\int_W \omega^n = \int_{\partial_+ W} \lambda \wedge (\mathrm{d}\lambda)^{n-1} - \int_{\partial_- W} \lambda \wedge (\mathrm{d}\lambda)^{n-1},$$

each integral is non-negative and the left hand side is positive. A compact Liouville cobordism carries interesting quantitative invariants such as symplectic volume, capacities, actions of closed characteristics on the boundary, etc. However in certain situations it is more appropriate to rule these out by attaching cylindrical ends to W, the object then becomes more topological. Explicitly, one uses the Liouville flow of X to get collar neighborhoods of $\partial_\pm W$ of the form $[1, 1+\epsilon] \times \partial_- W$ and $[1-\epsilon, 1] \times \partial_+ W$ on which λ reads $r\alpha_-$ and $r\alpha_+$, and then smoothly glue the cylindrical ends $]0,1] \times \partial_- W$ and $[1, +\infty[\times \partial_+ W$. The resulting object is called the *completion* of W: it has a complete Liouville vector field and it is of *finite type*, namely there is a proper function $\phi : W \to \mathbb{R}$ which, outside of a compact set of W, is Lyapunov and without critical points. The contact manifold at $\pm$ infinity $\partial_{\pm\infty} W$ is canonically defined as the orbit space of the Liouville trajectories near $\pm$infinity. Any choice of hypersurface transverse to X near $\pm$ infinity is a model for $\partial_{\pm\infty} W$ and provides a contact form for it. The general notion of (infinite type) Liouville structure is as follows: (W, ω, X) such that X is complete and there exists a proper function ϕ and a sequence $c_i \underset{i\to\pm\infty}{\longrightarrow} \pm\infty$ such that $X.\phi > 0$ on $\phi^{-1}(c_i)$.

A version of Moser stability holds in this context but requires some care. A Liouville homotopy $(\omega_t, \lambda_t, X_t)$, $t \in [0,1]$, is a smooth family of complete Liouville structures such that for all t there is a proper function ϕ_t and a sequence $c_i^t \underset{i\to\pm\infty}{\longrightarrow} \pm\infty$ such that for s close enough to t, $X_s.\phi_t > 0$ along $\phi_t^{-1}(c_i^t)$.

Proposition 1.2. *[CE12, proposition 11.8]*

If λ_t is a Liouville homotopy, then there exists an ambiant isotopy ϕ_t and a smooth family of functions h_t so that $\phi_t^* \lambda_t = \lambda_0 + \mathrm{d}h_t$.

The more naive notion of homotopy, simply a smooth family of Liouville structures, (i.e., remove "for s close enough to t" in the above) is not reasonable: it would make all Liouville manifold structures on $\mathbb{R}^{2n}$ homotopic (exercise 4).

In the sequel we will simply use the term *Liouville manifold* to mean a complete Liouville structure without concave end (i.e. the function ϕ is exhausting: proper and bounded below). A Liouville manifold may be of infinite type for topological reasons (for example, its homology could be infinite dimensional) but, as discovered by Mark McLean, this can also happen for purely symplectic reason: there is a Liouville manifold structure on $\mathbb{R}^8$ which is not of finite type [Sei08, theorem 7.1].

(Warning: the following contains self-advertisement) For a finite type Liouville manifold (W, ω, X, λ), the contact manifold at infinity $(\partial_\infty W, \xi)$ is well-defined by λ (this is the space of orbits of X at infinity, with induced $\xi = \ker \lambda$) but not by ω. It is proved in [Cou14], that on $\mathrm{T}^* \mathrm{L}(7,1)$, there is a function h (not compactly supported) such that the contact manifold at infinity of $\lambda_{\mathrm{can}} + \mathrm{d}h$ is diffeomorphic to $\mathrm{L}(7,2) \times \mathrm{S}^2$ (and this is not diffeomorphic to $\mathrm{S}\,\mathrm{T}^* \mathrm{L}(7,1) = \mathrm{L}(7,1) \times \mathrm{S}^2$). This can be interpreted as the fact that there are finite type Liouville structures that are connected in the space of Liouville structures but not in the space of finite type Liouville structures.

1.3 Weinstein structures

It is hard to understand Liouville structures because the dynamics of the Liouville vector field is uncontrolled, we tame it by requiring the existence of a Lyapunov function.

Definition 1.3. *A* weinstein structure *on a compact cobordism W is a triple (ω, X, ϕ) such that ϕ is a Morse function (constant on $\partial_- W$ and on $\partial_+ W$ without critical points on ∂W), (ω, X) is a Liouville structure and X is a pseudo-gradient vector field for ϕ (see exercise 5). Such a function ϕ is called ω-convex.*

Again, it is convenient to complete a Weinstein cobordism by attaching cylindrical ends, the Morse function ϕ can be extended to the completion as a proper Morse function without adding any critical point. In the case where $\partial_- W$, the resulting object will be called a finite type Weinstein manifold.

The examples of Liouville manifold given above $\mathbb{R}^{2n}$, $\mathrm{T}^* M$ are Weinstein manifolds and can be nicely characterized using this notion, see exercises 6 and 7.

Recall that the stable manifold theorem asserts that for each critical point p, there is a neighborhood U of p such that the subsets

$$W_X^{s/u}(U; p) = \{q \in U; \phi_X^t(q) \underset{t \to +/-\infty}{\longrightarrow} p\}$$

are smoothly embedded discs that meet transversely at p and whose tangent spaces at p are the stable and unstables subspaces of the linearized vector field. Assuming that X is complete, one can then use the flow of X to prove that the global stable and unstable manifolds

$$W_X^{s/u}(p) = \{q \in W; \phi_X^t(q) \underset{t \to +/-\infty}{\longrightarrow} p\}$$

are embedded submanifolds (not properly embedded though), diffeomorphic to euclidean spaces. The following proposition is fundamental.

Proposition 1.4. *Let p be a hyperbolic zero of a Liouville vector field X in a symplectic manifold (W, ω). The stable manifold of p is ω-isotropic and the unstable manifold of p is ω-coisotropic.*

Proof. We restrict to the case where X is a linear vector field in a symplectic vector space (E,ω) and we view X as an endomorphism of E. In this case the stable and unstable manifolds are vector subspaces and can be given explicitly:

$$E^{\pm} = \bigoplus_{\lambda \in Sp(X), \pm Re(\lambda)>0} \bigoplus_{k \geqslant 1} \ker(X - \lambda \, \mathrm{id})^k$$

the stable and unstables subspaces, we have $E = E^+ \oplus E^-$.

E^- is isotropic: for $u, v \in E^-$, $\omega(u,v) = e^{-t}\omega(e^{tX}u, e^{tX}v) \underset{t\to+\infty}{\longrightarrow} 0$. Note that this argument also works if X is a symplectic vector field, i.e. $X.\omega = 0$.

E^+ is coisotropic: the symplectic orthogonal $(E^+)^\omega$ is X-invariant because E^+ is X-invariant and ω is X-invariant up to scaling. Since ω is non-degenerate and E^- is isotropic, we have $E^- \cap (E^+)^\omega = 0$ and hence $(E^+)^\omega$ is contained in E^+. □

In particular, the critical points of an ω-convex Morse function have index $\leqslant n$, this is a strong topological restriction on Weinstein manifolds, as opposed to Liouville manifolds.

The stable manifolds of the critical points of a Weinstein structure are isotropic, and since they are tangent to X, the Liouville form λ vanishes on their tangent spaces and so they intersect the level sets of ϕ along isotropic submanifolds for the contact structure $\xi = \ker\lambda$. Conversely, as we will discuss now, we can build Weinstein cobordisms by attaching handles along isotropic spheres in the contact level sets.

1.4 Handle attachment

We start from the following data: a contact manifold (M,ξ), an isotropic sphere $j : \mathrm{S}^{k-1} \to M$ and a symplectic trivialization of the symplectic normal bundle of j, i.e. $(\mathrm{T}\,\mathrm{S}^{k-1})^\perp / \mathrm{T}\,\mathrm{S}^{k-1} \simeq \mathbb{C}^{n-k}$ (here $\perp$ refers to the canonical conformal symplectic structure on ξ). The following model is exposed in [Wei91].

Consider $\mathbb{R}^{2n} = \mathrm{T}^*\mathbb{R}^k \times \mathbb{C}^{n-k}$ with the symplectic form $\omega = \sum_{i=1}^k \mathrm{d}p_i \wedge \mathrm{d}q_i + \sum_{j=1}^{n-k} \mathrm{d}x_j \wedge \mathrm{d}y_j$ and the Liouville vector field $X = 2\sum_i p_i\partial_{p_i} - \sum q_i \partial_{q_i} + \frac{1}{2}\sum_j x_j\partial_{x_j} + y_j\partial_{y_j}$. The stable/unstable subspaces are $E^+ = \{q = 0\} \simeq \mathbb{R}^{2n-k}$ and $E^- = \{p = z = 0\} \simeq \mathbb{R}^k$. For $\epsilon > 0$, consider the region $H_\epsilon = \{|q| \leqslant 1, |p|^2 + |z|^2 \leqslant \epsilon\}$ which is a tubular neighborhood of the isotropic disc $D^k = \{p = 0, z = 0, |q| \leqslant 1\}$. Let us saturate H_ϵ by the Liouville flow to get M_ϵ, a "non-compact Morse model". Then $M_\epsilon \setminus E_+$ can be identified with the symplectization of a small neighbourhood of S^{k-1} in (M,ξ) via the choice of a Weinstein tubular neighborhood of S^{k-1}. This choice depends on parameters in a contractible space provided they are consistent with our choice of symplectic framing. Then one considers $\mathrm{S}_\xi\, M \cup M_\epsilon$ where we glue using the previous identification. One can then modify any section α of $\mathrm{S}_\xi M$ to make it coincide with $\{|p|^2 + |z|^2 = \epsilon\}$ near E_- and this will serve as top boundary for a Weinstein cobordism. One can also construct a function ϕ with pseudo gradient X and appropriate boundary conditions. As an application one can define the connect sum of two contact manifolds as the result of attaching a 1-handle. See exercise 8 for a handle attachment description of $\mathrm{T}^* \mathrm{S}^n$.

For $n = 2$, a finite type Weinstein manifold can be presented as one 0-handle, a bunch of 1-handles, and 2-handles. After attaching the 1-handles, the contact manifold is a connect sum of $\mathrm{S}^1 \times \mathrm{S}^2$ and there is a standard way to draw the attaching Legendrian spheres in this manifold. This makes the theory very combinatorial and computable in this case.

Also note that one can attach handles backwards, i.e. on the concave end. For that one needs to find a contact embedding of a neighborhood of the "coisotropic" sphere S^{2n-k-1}. But if $k < n$, it is harder to control such spheres and this is less used (unless (M, ξ) is overtwisted, as in [EM15]).

It is also possible to build Lagrangian submanifolds in Weinstein cobordisms by attaching handles. The main remark is that in the model handle $\mathrm{T}^* \mathbb{R}^k \times \mathbb{C}^{n-k}$, there are Lagrangian subcobordisms $\mathrm{T}^*_{\mathbb{R}^j} \mathbb{R}^k \times \mathbb{R}^{n-k} = \{q_{j+1} = \cdots = q_k = 0, p_1 = \cdots = p_j = 0, y = 0\}$ for $0 \leqslant j \leqslant k$ on which the function $\phi = |p|^2 - |q|^2 + |z|^2$ has a critical point of index j at 0. The Lagrangian cobordisms that admit such a Weinstein handle presentation are called *regular*, see [EGL15].

1.5 Hamiltonian dynamics and Weinstein structure

In view of computing symplectic homology, we explain how to construct linear at infinity Hamiltonians whose dynamics can be controlled using the handle presentation. The following statement should be checked more carefully.

Proposition 1.5. *Let (V, ω, X, ϕ) be a Weinstein manifold of finite type, $T > 0$ and α a contact form for $M = \partial_\infty V$ which has no periodic orbit of period T. There exists an exhausting function $H : V \to \mathbb{R}$ which is linear at infinity ($X.H = H$) and whose 1-periodic orbits are precisely the following:*

- *critical points of ϕ (a finite set).*
- *Reeb orbits of α of period $< T$.*

Moreover, one can impose the Conley-Zehnder index of the subcritical critical points of f to be arbitrary large.

Sketch of proof. We start from the function ϕ, in the region $\mathrm{S}\, M = M\times]0,+\infty[\subset V$, we deform the function ϕ (by pushing its level sets along the trajectories of X) so that its coincides with Tr near the region $\{r \geqslant 1\}$. By our assumption, there are no 1-periodic orbits of ϕ in this region, since $X_\phi = TR_\alpha$ there. Now we modify ϕ in the region $A = \{\phi \leqslant T\}$. For $\epsilon > 0$ sufficiently small, the function $\epsilon\phi$ has no 1-periodic orbit in the region A other than the critical points of ϕ. Near $r = 1$, $\phi = Tr$, and we interpolate between the function Tr and ϵTr by a smooth increasing function $h(r)$. We then get a function with the required properties (equal to $\epsilon\phi$ in A and equal to Tr near infinity).

For the subcritical critical points, let us assume that $\phi = a + p^2 - q^2 + z^2$ in an appropriate Darboux chart $\mathrm{T}^* \mathbb{R}^k \times \mathbb{C}^{n-k}$ with $k < n$ (this is possible after a deformation of f near the critical point). We replace ϕ by $a + p^2 - q^2 + c|z|^2$ near the critical point and leave ϕ unchanged out of a small neighborhood. For $c > 0$ very large, this will make the Conley-Zehnder index very large. □

For critical points of top index, there is no z coordinate to play with as in the above proof and the Conley-Zehnder index cannot be changed. In fact, these critical points survive in symplectic homology and are idempotent elements (reference ? Tobias' lectures ?).

The next goal is to understand the Reeb dynamics before and after Weinstein surgery. A contact form α is called *non-degenerate* if all its closed orbits are non-degenerate. For any $T > 0$, there is then finitely many Reeb orbits of period $< T$. If we are given an isotropic sphere S, we require further that the map $\mathbb{R} \times S \to M$ induced by the Reeb flow is transverse to S. In the subcritical case, this transversality means that there are no Reeb chords, while in the Legendrian case, this means that the Reeb chords are non-degenerate, and hence in finite number below any period T.

Proposition 1.6. *A generic contact form is non-degenerate.*

Here is the main statement, which is a building block in the work of Bourgeois, Ekholm and Eliashberg in [BEE12].

Theorem 1.7. *Let (M, ξ) be a contact manifold, S an isotropic sphere with trivialized symplectic normal bundle and α a non-degenerate contact form. Let (M', ξ') be the result of Weinstein surgery along S. For any $T > 0$, there is a contact form α' on M' with the following properties:*

- *If S is subcritical, the Reeb orbits of period $< T$ of α' are in bijection with those of α.*

- *If S is Legendrian, the Reeb orbits of period $< T$ of α' are in bijection with those of α + cyclic words in Reeb chords of S of total length $< T$.*

Remark 1.8. *It may be possible to deduce a simple proof of Cieliebak's theorem (see [Cie02]) that symplectic homology vanishes for subcritical Weinstein manifold from all this.*

1.6 Stein manifolds and affine varieties

Weinstein manifolds are the symplectic counterpart of Stein manifolds. These are complex manifolds that admit a proper holomorphic embedding in $\mathbb{C}^N$ for some N. For such a submanifold X the function $\phi = |z|^2$ is exhausting and strictly pluri-subharmonic (or i-convex), that is $-\mathrm{dd}^c\phi = 2i\mathrm{d}z \wedge \mathrm{d}\bar{z}$ is a symplectic form compatible with i. Conversely if a complex manifold (V, J) admits an exhausting J-convex function ϕ, then it admits a proper holomorphic embedding in some $\mathbb{C}^N$ by a difficult theorem of Grauert. So we may take the existence of ϕ as the definition of Stein manifolds. Moreover one can require ϕ to be Morse since Morse functions are C^∞-dense and J-convexity is preserved by a C^2-small deformation. To ϕ is associated a Weinstein structure $(\lambda_\phi = -\mathrm{d}^c\phi, \phi)$ (see exercise 9). Note however that the gradient vector field may not be complete but this can be arranged by composing by a sufficiently convex function $g : \mathbb{R} \to \mathbb{R}$ (see [CE12, proposition 2.11]). The ω-convex functions are the symplectic analogue of J-convex functions, and are much easier too work with (in a sense because they just involve a condition on the derivative not on the second derivative). It is a deep theorem of Cieliebak and Eliashberg (see [CE12]), building on earlier work of Eliashberg ([Eli90]), that every Weinstein structure can be deformed to one coming from a Stein structure.

An important class of examples of Stein manifolds are algebraic affine varieties. Let L be an ample holomorphic line bundle on a complex manifold X. Ampleness means that there exists a hermitian connection on L whose associated Chern connection has curvature $\kappa \in \Omega^2(X; i\mathbb{R})$ that writes $\kappa = -i\omega$ where ω is a symplectic form compatible with the complex structure on X. Take a holomorphic section s which vanishes transversely and denote $D = s^{-1}(0)$ (one can also consider more singular divisors D). Then the function $\phi = -\log|s|$ is an i-convex function on $X \setminus D$. Indeed, near a point of $X \setminus D$, one can pick a local holomorphic trivialization where $s = 1$, and the hermitian metric writes $e^{-\phi}|.|$. Then the curvature κ of the Chern connection is $-2\partial\overline{\partial}\phi$ (this follows from the definition of the Chern connection), writing $\kappa = i\omega$ we get $\omega = 2i\partial\overline{\partial}\phi$ is a symplectic form compatible with i (by definition of ampleness). But $-\mathrm{dd}^c = 2i\partial\overline{\partial}$, so the claim follows. Moreover we have some control on the critical points of ϕ (see the end of [McL09, section 2]).

Lemma 1.9. *There exists a compact set K of $X \setminus D$, so that ϕ has no critical points outside of K.*

One can cook from this a more or less canonical finite type Weinstein manifold structure on $X \setminus D$. For more on this topic, see [McL09]. The examples leading to theorem 1.13 come from such constructions.

1.7 Flexibility

A Weinstein structure is determined by all the attaching spheres of the critical points (see exercise 10). One would need some results about the classification of isotropic spheres up to isotopy. In the *subcritical* case (i.e., not Legendrian), Gromov proved that the h-principle holds. This roughly means that the only obstructions for existence or isotopy of isotropic spheres are topological, there is no symplectic rigidity phenomena. However, in the Legendrian case, there is symplectic rigidity: the Bennequin inequalities are the first instance of this phenomenon, they give constraints on the classical invariants of a Legendrian knot in $\mathbb{R}^3$. Eliashberg and Chekanov also discovered Legendrian knots with the same classical invariants but still not Legendrian isotopic.

In [Eli90], Eliashberg introduced a stabilization procedure for Legendrian submanifolds which enabled him to prove that Legendrian spheres exist in every formal class. This is the key to the following result:

Theorem 1.10 (Eliashberg)**.** *Let W be a manifold of dimension $2n \geqslant 6$ with an exhausting Morse function ϕ with critical points of index $\leqslant n$ and a non-degenerate two-form η. There is a Weinstein structure (ω, X, ϕ) where ω is homotopic to η in the space of non-degenerate 2-forms.*

This is wrong in dimension 4 because of Bennequin's inequality.

Much later, Murphy discovered that a full h-principle holds in dimension $\geqslant 5$ for a class of Legendrian submanifolds called *loose* (see [Mur12]). This result in particular reproves by a completely different method the existence result of Eliashberg for Legendrian spheres, in fact the stabilization procedure produces loose Legendrian submanifolds.

Definition 1.11. *A Weinstein manifold (W, ω) is flexible if there exists an excellent* [1] *ω-convex function such that all attaching spheres of critical points of index n are loose (in a level set just below the corresponding critical level set).*

[1] excellent = Morse and with at most one critical point in each level set.

There is deep subtlety in this definition. Murphy and Siegel discovered that this actually depends on the function ϕ: on a flexible Weinstein manifold there always exist excellent ω-convex functions ϕ which do not satisfy the above (see [**?**]).

Subcritical Weinstein structure are flexible, and Eliashberg and Gromov already proved some flexibility results for subcritical Weinstein structures in [EG91]. This work has then been developped by Cieliebak and Eliashberg in [CE12] culminating in the following theorem.

Theorem 1.12 (Cieliebak-Eliashberg)**.** *Let W be a manifold of dimension $2n \geqslant 6$ which admits an exhausting Morse function with critical points of index $\leqslant n$ and a non-degenerate 2-form η. Then there is a unique up to Weinstein homotopy flexible Weinstein structure on W in the formal class η.*

In addition to Gromov and Murphy's h-principle for isotropic spheres, the above theorem relies on non-trivial results from Cerf theory, specifically the fact that the space of functions with critical points of index $\leqslant n$ is connected.

Not every Weinstein manifold is flexible, for example we have:

Theorem 1.13 (McLean [McL09])**.** *There exists infintely many non symplectomorphic Weinstein structures on $\mathbb{R}^{2n}$ $n \geqslant 4$.*

Subsequent work of Abouzaid-Seidel and Bourgeois-Ekholm-Eliashberg shows that this exotica phenomenon holds much more generally in every dim $\geqslant 6$ (see [CE12, theorem 17.2]).

1.8 Exercises

Exercise 1 (read on Chris Wendl's blog)**.** *Let W be a symplectic manifold of dimension $2n \geqslant 4$ and $\Sigma \subseteq W$ a cooriented closed hypersurface. Prove that Σ cannot be both ω-convex and ω-concave. Notice that this is wrong for $n = 1$.*

Hint: if λ_+ and λ_- are convex and concave Liouville forms for Σ, then $\beta = (\lambda_+ - \lambda_-) \wedge \omega^{n-1}$ is a volume form but is also exact $\beta = \mathrm{d}((\lambda_+ - \lambda_-) \wedge \lambda_+ \wedge \omega^{n-2})$.

Exercise 2. *Check the computations of the Reeb flow on the unit cotangent bundle and obtain the second order equation for geodesics.*

Exercise 3. *Prove that the Hamiltonian vector field of a function $H : \mathrm{S}_\xi M \to \mathbb{R}$ satisfying $X.H = H$ lifts a contact vector field. Describe it in a trivialization.*

Exercise 4. *Prove that for the naive notion of Liouville homotopy, all Liouville manifold structures on $\mathbb{R}^{2n}$ are homotopic.*

Hint: Let λ is a Liouville manifold structure on $\mathbb{R}^{2n}$. First, by adding to λ the differential of a function supported in a neighborhood of zero, arrange that $\lambda = \frac{1}{2} r^2 \mathrm{d}\theta$ near 0. Then for $t \in [0, 1[$, define λ_t as the pullback λ by $z \mapsto (1-t)z$ and extend by $\lambda_1 = \frac{r^2 \mathrm{d}\theta}{2}$.

Exercise 5. *Let ϕ be a Morse function and X a vector field satisfying $X.\phi > 0$ away from critical points. Prove that X vanishes at critical points and that it has hyperbolic zeroes (i.e. the eigenvalues of the linearized vector field have non zero real part) if and only if the function $X.\phi$ has a non-degenerate minimum at each critical point. Such a vector field is called a* pseudo-gradient *vector field for ϕ.*

Exercise 6. *Let (W, ω) be a symplectic manifold of dimension $2n$ which admits an exhausting ω-convex function ϕ with just one critical point. Prove that W is exact symplectomorphic to $\mathbb{R}^{2n}$.*

Exercise 7. *Let (W, ω) be a symplectic manifold which admits an exhausting ω-convex Morse function ϕ and a Liouville pseudo-gradient X whose skeleton is an embedded Lagrangian submanifold L. Prove that W is exact symplectomorphic to $\mathrm{T}^* L$.*

Exercise 8. *Let S^{2n-1} be endowed with the standard contact structure and $j : \mathrm{S}^{n-1} \to \mathrm{S}^{2n-1}$ the standard Legendrian unknot. Prove that (-1)-surgery on S^{2n-1} along j produces a contact manifold contactomorphic to the sphere cotangent bundle of S^n with its standard contact structure.*

Exercise 9 (Stein vs Weinstein)**.** *Let (V, J) be an almost-complex manifold. A function $\phi : V \to \mathbb{R}$ is called J-convex if $-\mathrm{dd}^c\phi(v, Jv) > 0$ for all $v \neq 0$, where $\mathrm{d}^c\phi = \mathrm{d}\phi \circ J$.*

1. *Prove the identity: for any smooth function ϕ,*

$$-\mathrm{dd}^c\phi(v, Jw) + \mathrm{dd}^c\phi(w, Jv) = \mathrm{d}\phi(N_J(v, w)),$$

   where $N_J(v, w) = [v, w] - [Jv, Jw] + J[Jv, w] + J[v, Jw]$ is the Nijenhuis tensor.

2. *Conclude that, if J is integrable (i.e., $N_J = 0$), and ϕ is J-convex, $\omega_\phi := -\mathrm{dd}^c\phi$ is a symplectic form compatible with J.*

3. *Prove that the gradient of ϕ for the metric $g(v, w) = \omega(v, Jw)$ is equal to the Liouville vector field dual to $-\mathrm{d}^c\phi$.*

Exercise 10. *Let (W, ω) be a Weinstein manifold, ϕ an excellent ω-convex function, and p a critical point of ϕ. Consider a regular level set $N = \phi^{-1}(c)$ so that $c < \phi(p)$ and $\phi^{-1}[c, \phi(p)]$ contains just one critical point, and the attaching sphere $S \subset N$ corresponding to a Liouville pseudo-gradient X. Prove that for any isotropic isotopy $(S_t)_{t\in[0,1]}$ of $S = S_0$, after a deformation of ϕ among Lyapunov functions for X, there is a homotopy $(X_t)_{t\in[0,1]}$ of Liouville pseudo-gradients for ϕ that induces S_t.*

2 Lefschetz fibrations and open book decompositions

2.1 lefschetz pencils in algebraic geometry

A *hyperplane pencil* in $\mathbb{C}\mathrm{P}^n$ is the set of hyperplanes containing a given codimension 2 projective subspace B'. We can describe it as a projective map $\mathbb{C}\mathrm{P}^n \setminus B' \to \mathbb{C}\mathrm{P}^1$. Given a smooth projective variety X in $\mathbb{C}\mathrm{P}^n$, we say that a hyperplane pencil is a *Lefschetz pencil* for X if it is in general position with respect to X, namely:

1. the base locus B' is transverse to X,
2. the holomorphic map $f : X \setminus B \to \mathbb{C}\mathrm{P}^1$, where $B = B' \cap X$, has only non-degenerate critical points,
3. all critical values are distinct.

From the holomorphic Morse lemma we get, for each critical point p of f, holomorphic coordinates $(z_1, \dots, z_n)$ centered at p and w centered at $f(p)$ where $f = z_1^2 + \dots + z_n^2$. The issue that f is not defined on B can be resolved in two ways:

- Blow-up B and get a map $\widetilde{X} \to \mathbb{C}\mathrm{P}^1$ where $\widetilde{X}$ is the blow-up of X along B (namely, replace B by its projectivized normal bundle).
- Remove a regular fiber and get a map $W \to \mathbb{C}$ where $W = X \setminus (f^{-1}(\infty) \cup B)$.

For example take the Veronese embedding $\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^5$ given by $[x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : yz : zx]$. A hyperplane pencil on $\mathbb{C}\mathrm{P}^5$ induces a pencil of conics on $\mathbb{C}\mathrm{P}^2$. If it is a Lefschetz pencil, then the base locus B is 4 points in general position and the pencil is defined as the set of conics passing through these points. Exactly three fibers are singular: they are union of two lines passing through B. Removing a fiber one gets a Lefschetz fibration $W \to \mathbb{C}$ with fiber a 4-punctured sphere and three critical values.

Another example is the Segre embedding $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^3$ given by $([x_1 : y_1], [x_2 : y_2]) \mapsto [x : y : z : t] = [x_1x_2 : x_1y_2 : y_1x_2 : y_1y_2]$ whose image is the quadric surface $\{xt = yz\}$. By removing a regular fiber, we get a Lefschetz fibration $f : W \to \mathbb{C}$ with fiber $\mathrm{T}^* \mathrm{S}^1$ and 2 critical points.

One can endow the total space W with a complete finite type Liouville structure and in the first example we get $\mathrm{T}^* \mathbb{R}\mathrm{P}^2$ while in the second we get $\mathrm{T}^* \mathrm{S}^2$.

2.2 Lefschetz fibrations on finite type Liouville manifolds

Lefschetz fibrations were introduced in symplectic geometry in the work of Donaldson, who proved a remarkable existence result on compact symplectic geometry (see [Don99]). To do this he invented so-called *approximately holomorphic* techniques, and this proved useful for other applications later (see theorem 2.4 and 2.6). Here we focus on the case of finite type Liouville manifold: we want to present them as the total space of a *Lefschetz fibration* over $\mathbb{C}$ with fiber a finite type Liouville manifold. Such a structure induces an *open book decomposition* of its contact boundary. A good reference is [Sei03, section 1].

Definition 2.1. *Let (W, ω, X, λ) be a finite type Liouville manifolds. A* Lefschetz fibration *on W is a map $f : W \to \mathbb{C}$ satisfying the following properties:*

1. *(Triviality near the horizontal boundary)*

   There exists a contact manifold (B, ξ), an open set $U \subseteq W$ such that $f : W \setminus U \to \mathbb{C}$ is proper and a codimension zero embedding $\Phi : U \to \mathrm{S}_\xi B \times \mathbb{C}$ such that $\mathrm{pr}_2 \circ \Phi = f$ and $\Phi_* \lambda = \mathrm{pr}_1^* \lambda_\xi + \mathrm{pr}_2^* \mu$ where $\mu = \frac{1}{2} r^2 \mathrm{d}\theta$.

2. *(Lefschetz type critical points)*

   There are only finitely many points where $\mathrm{d}f$ is not surjective and for any such critical point p complex Darboux coordinates $(z_1, \dots, z_n)$ centered at p so that $f(z_1, \dots, z_n) = f(p) + z_1^2 + \cdots + z_n^2$. Moreover, there is at most one critical point in each fiber of f.

3. *(Transversality to the vertical boundary)*

   There exists $R > 0$ such that X lifts the vector field $\frac{1}{2} r \partial_r$ near the region $\{|f| \geqslant R\}$.

4. *(Symplectic fibers)*

   Away from the critical points, ω is non-degenerate on the fibers of f.

This is hard to digest, so we now make a long series of remarks about this definition. Denote by $\mathrm{crit}(f)$ the set of critical points of f and by $\mathrm{vcrit}(f)$ the set of critical values. By assumption the map $f : \mathrm{crit}(f) \to \mathrm{vcrit}(f)$ is a bijection.

Symplectic connection On $W \setminus \mathrm{crit}(f)$, denote by $V = \ker \mathrm{d}f$ the vertical subbundle of $\mathrm{T}\, W$ and by H the subbundle ω-orthogonal to V. We have the ω-orthogonal decomposition

$$\mathrm{T}\, W = V \oplus H.$$

So H is a connection in the sense of Ehresmann. We claim that the parallel transport maps are well-defined. If $\gamma : [0,1] \to \mathbb{C} \setminus \operatorname{vcrit}(f)$ is a smooth path, for each $p \in f^{-1}(\gamma(0))$, we lift it as a horizontal path $\widetilde{\gamma} : [0,1] \to W$ starting from W, namely $\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}t} \in H$, and declare $\widetilde{\gamma}(1)$ to be the image of p. However it is not clear the the path $\widetilde{\gamma}$ can be defined up to $t = 1$, but this is ensured by the first condition: in the trivialization Φ, the decomposition $V \oplus H$ coincides with $\mathrm{T}\,\mathrm{S}_\xi\, B \oplus \mathrm{T}\,\mathbb{C}$ because the Liouville form λ splits as $\mathrm{pr}_1^* \lambda_\xi + \mathrm{pr}_2^* \mu$, hence when $\widetilde{\gamma}$ enters U, then $\mathrm{pr}_1 \circ \Phi \circ \widetilde{\gamma}$ is constant and $\widetilde{\gamma}$ cannot escape to infinity. Moreover, the parallel transport maps are exact symplectomorphisms in the following strong sense: if $\phi : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$ is the parallel transport map, we have $\phi^*\lambda = \lambda + \mathrm{d}k$ where k is a smooth function which vanishes outside of a compact set. Indeed, let $w \in \mathbb{C}$, $\widetilde{\nu}$ be the horizontal lift of a vector field ν defined near w, and $i_w : f^{-1}(w) \to W$ the inclusion, we have:

$$i_w^*(\widetilde{\nu}.\lambda) = i_w^*(\widetilde{\nu} \lrcorner \omega) + i_w^*(\mathrm{d}(\widetilde{\nu} \lrcorner \lambda)) = \mathrm{d}h$$

where $h = i_w^*(\widetilde{\nu} \lrcorner \lambda)$ is not necesseraly compactly supported but it is constant at infinity by inspection in the trivialized region U. One can then subtract this constant and the claim follows by integrating this equation. Hence regular fibers are all exact symplectomorphic, we pick one and call it $(F, \omega_F, X_F, \lambda_F)$. We have $\partial_\infty F = B$ and a canonical embedding $\mathrm{S}_\xi\, B \to F$ near the convex end, so we may equivalently let the trivialization Φ take values in $F \times \mathbb{C}$.

The map $f : f^{-1}(\mathbb{C} \setminus \operatorname{vcrit}(f)) \to \mathbb{C} \setminus \operatorname{vcrit}(f)$ inherits the structure of a locally trivial fibration with fiber F and structure group

$$\mathrm{Symp}^e(F) = \{\phi \in \mathrm{Diff}(F) | \phi^*\lambda_F = \lambda_F + \mathrm{d}h, \phi = \mathrm{id} \text{ and } h = 0 \text{ outside of a compact set}\}.$$

To find a local trivialization of this bundle around $w \in \mathbb{C} \setminus \operatorname{vcrit}(f)$, consider the rays starting from w and lift them horizontally, this allows to define a map $\Phi : F \times \mathrm{Op}(w) \to W$ with $f \circ \Phi = \mathrm{pr}_2$ and so that the lifted rays are the obvious ones in $F \times \mathrm{Op}(w)$. Since parallel transport maps are exact symplectomorphisms, we automatically have

$$\Phi^*\lambda = \mathrm{pr}_1^* \lambda_F + \mathrm{d}R + \nu$$

where R is a function which vanishes near the horizontal boundary and ν is a 1-form which vanishes on vertical vectors and writes $\mathrm{pr}_2^* \mu$ near the horizontal boundary. Using polar coordinates (r, θ) centered at w the 1-form ν can be written $\nu = K\mathrm{d}r + H\mathrm{d}\theta$ where K and H are functions of (r, θ) near the horizontal boundary. The functions K and H should be thought of as Hamiltonians generating the parallel transport maps along the r and θ directions. In fact, since we have trivialized f along rays from w, the function K is constant on each fiber everywhere (not just near the horizontal boundary). We will come back to this kind of computations when discussing the symplectic Dehn twist.

Thurston's trick If one has a manifold W' with a 1-form λ (but $\omega = \mathrm{d}\lambda$ not necessarily symplectic) and a map $f : W' \to \mathbb{C}$ satisfying 2. and 4. and the following modified 1.

1'. There exists an open set $U \subseteq W$ such that $f : W \setminus U \to \mathbb{C}$ is proper and a codimension 0 embedding $\Phi : U \to F \times \mathbb{C}$ such that $\mathrm{pr}_2 \circ \Phi = f$ and $\Phi_* \lambda = \mathrm{pr}_1^* \lambda_F$.

then one can construct a complete finite type Liouville manifold W (diffeomorphic to W') with a map still denoted $f : W \to \mathbb{C}$ which is a Lefschetz fibration with same fiber as $f : W' \to \mathbb{C}$. This is a version of Thurston's trick as in [Thu76]. The starting point is to consider the 1-form $\lambda_k = \lambda + kf^*\mu$ where $\mu = \frac{1}{2}r^2\mathrm{d}\theta$. Near the critical points, there are complex coordinates in which $f = f(p) + z_1^2 + \cdots + z_n^2$ and $\mathrm{d}\lambda = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$. Since f is holomorphic $f^*\mathrm{d}\mu(v, Jv) = \mathrm{d}\mu(\mathrm{d}f(v), i\mathrm{d}f(v)) \geqslant 0$ and hence $\mathrm{d}\lambda + kf^*\mathrm{d}\mu$ is symplectic. Away from critical points, we have the decomposition $\mathrm{T}\,W = V \oplus H$ which is still orthogonal for $\mathrm{d}\lambda_k$. Observe that near the critical points, $\mathrm{d}f : H \to \mathrm{T}\,\mathbb{C}$ is orientation preserving, so $f^*\mathrm{d}\mu$ is positive on H. Hence there exists $K > 0$ such that for any $k \geqslant K$, $\mathrm{d}\lambda_k$ becomes positive on H everywhere, and since it is unchanged on V, it is a symplectic form. Now let us compute the corresponding Liouville vector field X_k for $k \geqslant K$ and away from critical points. Denote by $Z = \frac{1}{2}r\partial r$ the Liouville vector field of μ (notice this is the same for $k\mu$) and by $\widetilde{Z}$ its horizontal lift. We have $\widetilde{Z} \lrcorner (k\pi^*\mathrm{d}\mu) = k\pi^*\mu$. Denoting by π_V and π_H the projections to V and H, there is a vertical vector T such that $T \lrcorner \mathrm{d}\lambda = \lambda \circ \pi_V$ (because $\mathrm{d}\lambda$ is non-degenerate on V). Since $\lambda \circ \pi_H - \widetilde{Z} \lrcorner \mathrm{d}\lambda$ vanishes on V, there is an horizontal vector Y_k such that $Y_k \lrcorner \mathrm{d}\lambda_k = \lambda \circ \pi_H - \widetilde{Z} \lrcorner \mathrm{d}\lambda$. Finally $X_k = \widetilde{Z} + T + Y_k$ is the vector field that we are looking for:

$$\begin{aligned} X_k \lrcorner (\mathrm{d}\lambda + k\pi^*\mathrm{d}\mu) &= \widetilde{Z} \lrcorner \mathrm{d}\lambda + k\pi^*\mu + \lambda \circ \pi_V + \lambda \circ \pi_H - \widetilde{Z} \lrcorner \mathrm{d}\lambda \\ &= \lambda + k\pi^*\mu. \end{aligned}$$

The key point now is that $Y_k \underset{k \to +\infty}{\longrightarrow} 0$, so for any $R > 0$ such that $\mathrm{vcrit}(f) \subset \{|w| < R\}$, and k sufficiently large, X_k is transverse to the hypersurface $M_R = \{|f| = R\}$. The last step to construct W is to complete $\{|f| \leqslant R\}$ in the horizontal direction and extend the map f. Denote by α_k the contact form induced by λ_k on M_R and glue $[1, +\infty[\times M_R$ to $\{|f| \leqslant R\}$ as we did for the completion of a Liouville domain (recall this is a canonical procedure), this is our manifold W. The Liouville form λ_k smoothly extends as $u\alpha_k$ in the region $[1,+\infty[\times M_R$, where u is the coordinate in $[1,+\infty[$. Finally we extend f to $[1,+\infty[\times M$ by: $f(u,p)=\sqrt{u}f(1,p)$ (this actually needs to be smoothed near $M_R\setminus U$). This does not add any critical point and the fiber over $re^{i\theta}$ for $r\geqslant R$ is symplectic for it is the image of the fiber over $Re^{i\theta}$ by the Liouville flow at time $2\ln(r/R)$. The last thing we need to check is that the trivialization near the horizontal boundary can be extended over the whole $\mathbb{C}$. This is done by horizontally lifting rays from the origin and extending the trivialization. Explictly, in the region $[1,+\infty[\times M_R$, the horizontal subbundle H is spanned by ∂_θ and $\partial_u-\frac{X_F}{u}$. Hence we extend the embedding $U\to F\times\mathbb{C}$ by $(u,\theta,q)\mapsto(\phi_{X_F}^{\ln(u)}(q),\sqrt{u}Re^{i\theta})$ and check:

$$\Phi^*(\lambda_F+\frac{kr^2}{2}\mathrm{d}\theta)=u(\lambda_F+\frac{kR^2}{2}\mathrm{d}\theta).$$

Finally, compose f by $w\mapsto\frac{w}{\sqrt{k}}$ to exactly match condition 1. of definition 2.1.

2.3 The boundary of a Lefschetz fibration as an open book

Let (W,ω,X,λ) be a complete finite type Liouville manifold and $f:W\to\mathbb{C}$ be a Lefschetz fibration with fiber $(F,\omega_F,X_F,\lambda_F)$. We explain now how this induces a natural decomposition of the ideal boundary $M=\partial_\infty(W,\lambda)$. Take $R>0$ sufficiently large so that $\mathrm{vcrit}(f)\subset\{f<R\}$ and X lifts $\frac{1}{2}r\partial_r$ near $\{f\geqslant R\}$. Consider also a trivialization $\Phi:U\to F\times\mathbb{C}$ near the horizontal boundary and a contact form α on $\partial_\infty F$ so that $F\times[1,+\infty[\times D_R\subset\Phi(U)$ where $[1,+\infty[\times F\subset F$ is the cylindrical end where $\lambda_F=u\alpha$. Then this exhibits M as the union of two pieces:

- (Suspension of a page) $\{|f|=R\}$ with contact form λ,
- (Neighborhood of the binding) $\{u\circ\Phi=1\}$ with contact form $\alpha+\frac{1}{2}r^2\mathrm{d}\theta$.

These two pieces are naturally glued in M by the Liouville flow. The central fiber $f^{-1}(0)$ intersects M in the closed codimension 2 submanifold $B=\partial_inftyF$ and the map $\theta=\arg(f):\{f=R\}\to\mathbb{R}/2\pi\mathbb{Z}$ is actually well-defined as a map $M\setminus B\to\mathbb{R}/2\pi\mathbb{Z}$ since X lifts $\frac{1}{2}r\partial_r$ in $\{|f|\geqslant R\}$.

Definition 2.2. *An* open book decomposition *of a closed manifold M is a closed codimension 2 submanifold B with trivial normal bundle together with a fibration $\theta:M\setminus B\to\mathbb{R}/2\pi\mathbb{Z}$ which is equal to the angle coordinate in some tubular neighborhood $D^2\times B$ of B.*

B is called the *binding* and the fibers of θ (or rather their closures in M) are called the pages of the open book. Such topological decomposition have been extensively studied. For example it is known that every oriented odd-dimensional manifold admits such a decomposition (Alexander in dimension 3, Lawson and Quinn in dimension $\geqslant 5$). Their importance in contact geometry has been discovered by Giroux (see [Gir02]), who made the following definition

Definition 2.3. *Let M be a closed manifold. A contact structure ξ is carried by an open book decomposition (B, θ) if there exists a contact form α with the following properties:*

- *α is a positive contact form on B,*
- $\mathrm{d}\alpha$ *is a positive symplectic form on the pages,*
- *the orientations of B coming from α and from the oriented pages agree.*

Such a contact form is said to be adapted *to the open book.*

and proved the following theorem.

Theorem 2.4 (Giroux)**.** *On a closed manifold, every contact structure is carried by an open book.*

While in dimension 3 there is a combinatorial proof, the higher dimensional case relies on Donaldson's so-called *approximately holomorphic geometry.*

Let us come back to our discussion of Lefschetz fibrations. We claim that (B, θ) as defined just before definition 2.2 is an open book decomposition which carries ξ. One simply needs to construct an adapted contact form, there are infinitely many ways to do it, here is one. The contact form on $\{|f| = R\}$ writes $\frac{R^2}{r^2}(\alpha + \frac{kr^2}{2}\mathrm{d}\theta)$ because $2\ln(R/r)$ is the time that it takes to flow from $\{\Phi \circ u = 1\}$ to $\{|f| = R\}$. Pick a function $\rho(r)$ which satisfies $\rho(r) = \frac{R^2}{r^2}$ near $r = R$, $\rho(r) = 2 - r^2$ near 0 and with $\rho' < 0$ away from zero, and consider $\alpha_\rho = \rho(r)(\lambda_F + \frac{kr^2}{2}\mathrm{d}\theta)$. This smoothly glues with the contact form on $\{|f| = R\}$ and is adapted to the open book decomposition. The contact form α_ρ endows the pages $\{\theta = 0\}$ with a Liouville domain structure. In fact, the contact form induced by λ on $\{|f| = R\}$ is exact symplectomorphic to F and in a sense more canonical, but it converges to $+\infty$ on the binding.

To finish the discussion of the contact boundary, we discuss the monodromy of the open book. Let F be the fiber over $R \in \mathbb{C}$. The parallel transport around the loop Re^{it}, $t \in [0, 2\pi]$ gives an element $m \in \mathrm{Symp}^e(F)$, this is called the *monodromy* of the open book. Its conjugacy class in $\pi_0\mathrm{Symp}^e(F)$ determines the contact manifold up to contactomorphism.

Construction of a contact manifold from a symplectic open book There are several ways to proceed, here is one way. Consider a complete finite type Liouville manifold (F, λ) together with a symplectic diffeomorphism ϕ which is the identity outside of a compact set.

Lemma 2.5. *There is family $(\phi_t)_{t\in[0,1]}$ of symplectic diffeomorphisms equal to the identity outside of a compact set, and a positive function $h : F \to \mathbb{R}$ equal to 2π outside of a compact set such that $\phi_0 = \phi$ and $\phi_1^*\lambda = \lambda + \mathrm{d}h$.*

Proof. We have $\phi^*\lambda = \lambda + \beta$ where β is a compactly supported closed 1-form. The vector field Z defined by $Z \lrcorner \mathrm{d}\lambda = -\beta$ is a symplectic vector field and it vanishes outside of a compact set, so it is complete. One computes that $(\phi_Z^t)^*\beta = \beta$ and $(\phi_Z^t)^*\lambda - t\beta + \mathrm{d}k_t$ for some compactly supported functions k_t.

Then $\psi_t = \phi \circ \phi_Z^t$ satisfies $\psi_0 = \phi$ and

$$\psi_1^*\lambda = (\phi_Z^1)^*(\lambda + \beta) = \lambda - \beta + \mathrm{d}k_1 + \beta = \lambda + \mathrm{d}k_1.$$

At this point, one could take $\psi_t = \phi_t$ and $h = k_1 + 2\pi$ but the only issue is that h need not be positive. We arrange this by a second deformation as follows. Consider the Liouville vector field X dual to λ and, for $T \in \mathbb{R}$, $\theta_T = \phi_X^{-T} \circ \psi_1 \circ \phi_X^T$. We have

$$\theta_T^*\lambda = (\phi_X^T)^*(\psi_1^*(e^{-T}\lambda)) = e^{-T}(\phi_X^T)^*(\lambda + \mathrm{d}k_1) = \lambda + e^{-T}\mathrm{d}(k_1 \circ \phi_T).$$

For $T > 0$ large enough, $\sup |e^{-T}k_1 \circ \phi^T| < 2\pi$ as required. □

We assume that ϕ has been deformed using the lemma so that $\phi^*\lambda = \lambda + \mathrm{d}h$ with $h > 0$ and $h = 2\pi$ outside of a compact set.

Consider the manifold $\mathbb{R} \times F$ with the contact structure $\xi = \ker(\mathrm{d}\theta + \lambda)$ and the $\mathbb{Z}$-action generated by

$$\psi : (\theta, x) \mapsto (\theta - h(x), \phi(x)).$$

One checks that this action preserves the contact structure: $\psi^*(\mathrm{d}\theta + \lambda) = \mathrm{d}\theta - \mathrm{d}h + \lambda + \mathrm{d}h = \mathrm{d}\theta + \lambda$, and that the action is free and proper (this is the place where we use $h > 0$). Hence the quotient space M_ϕ is a contact manifold, actually the contact form $\mathrm{d}\theta + \lambda$ descends to the quotient, and moreover the end of this contact manifold is canonically identified with that of $\mathbb{R}/2\pi\mathbb{Z} \times F$ because $\phi = \mathrm{id}$ and $h = 2\pi$ outside of a compact set. It remains to close the manifold M_ϕ by adding the binding $B = \partial_\infty F$. We want a neighborhood of the binding to be contactomorphic to $\mathrm{D}^2 \times B$ with contact structure $\ker(r^2\mathrm{d}\theta + \alpha)$ where α is a contact form for B. For $\epsilon > 0$ sufficiently small we have a contact embedding $j_\epsilon : D_\epsilon^2 \setminus \{0\} \times B \to M_\phi$ given by

$j_\epsilon(r,\theta,x) = (\theta, i(\frac{1}{r^2}, x))$ where $i :]0,+\infty[\times B \to F$ is the trivialization of $\mathrm{S}\, B$ induced by α composed with the inclusion in F: indeed

$$j_\epsilon^*(\mathrm{d}\theta + \lambda) = \mathrm{d}\theta + \frac{\alpha}{r^2}$$

We can now form:

$$\overline{M_\phi} = (\mathrm{D}^2_\epsilon \times B) \bigcup_{j_\epsilon} M_\phi.$$

The first summand is a neighborhood of the binding while the second is the suspension of the symplectic diffeomorphism ϕ, and they are glued together by a contactomorphism between open subsets, so the result is a smooth contact manifold.

2.4 Critical points and vanishing cycles

So far we have mainly been concerned with the structure at infinity in a Lefschetz fibration. It is high time we dicuss the interior and especially the Lefschetz critical points. We closely follow [Sei03].

Radial trivialization and monodromy Let $f : \mathbb{C}^n \to \mathbb{C}$ be the map $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$. Endow $\mathbb{C}^n$ with the standard Liouville form $\lambda = \frac{1}{2}\sum r_i^2 \mathrm{d}\theta_i$. First we introduce the cotangent space of the $(n-1)$-sphere as

$$\mathrm{T}^*\,\mathrm{S}^{n-1} = \{(p,q) \in \mathbb{R}^n \times \mathbb{R}^n | |q| = 1, \langle p, q\rangle = 0\},$$

in which the canonical 1-form writes $\lambda_{\text{can}} = \sum_{i=1}^n p_i \mathrm{d}q_i$ but also $\lambda_{\text{can}} = \frac{1}{2}\sum(p_i\mathrm{d}q_i - q_i \mathrm{d}p_i)$ since $\langle p, q\rangle = 0$. For $s > 0$, the fiber $f^{-1}(s)$ is given by the equations $|x|^2 - |y|^2 = s, \langle x, y\rangle = 0$ and the map $\Phi_s : f^{-1}(s) \to \mathrm{T}^*\,\mathrm{S}^{n-1}$ given by

$$\Phi_s(x+iy) = (-|x|y, \frac{x}{|x|})$$

is a diffeomorphism and satisfies $\Phi_s^*\lambda_{\text{can}} = \lambda$:

$$\Phi_s^*\lambda_{\text{can}} = -\frac{1}{2}\sum y_i \mathrm{d}x_i + \frac{1}{2}\sum x_i \mathrm{d}y_i - \sum x_i y_i \left(-|x|\mathrm{d}(\frac{1}{|x|}) + \frac{1}{|x|}\mathrm{d}(|x|)\right) = \lambda.$$

For $s = 0$, this map is still well-defined and gives a diffeomorphism $\Phi_0 : f^{-1}(0)\backslash\{0\} \to \mathrm{T}^*\,\mathrm{S}^{n-1} \setminus \mathrm{S}^{n-1}$.

We claim that the parallel transport maps are well-defined. First observe that the horizontal distribution at $z \neq 0$, is spanned over $\mathbb{C}$ by $(\bar{z_1}, \ldots, \bar{z_n})$: for $v \in$

$\mathbb{C}^n$,$\mathrm{d}f(v) = f'(z)\mathrm{d}z(v) = 2\sum z_i v_i = 2h(\bar{z}, v) = 2(\omega(i\bar{z}, v) + i\omega(\bar{z}, v))$ where $h(u,v) = \sum \bar{u}_i v_i$ is the standard hermitian metric. Now we introduce the magical function $\kappa(z) = |z|^4 - |f(z)|^2$ which is horizontal: recall first that a vector $v \in \mathbb{C}^n$ can be written in complex coordinates $v = v\partial_z + \bar{v}\partial_{\bar{z}}$ and compute:

$$\mathrm{d}_z\kappa = 2|z|^2(\sum_i z_i \mathrm{d}\bar{z}_i + \bar{z}_i \mathrm{d}z_i) - 2f(z)\sum \bar{z}_i \mathrm{d}\bar{z}_i - 2\overline{f(z)}\sum z_i \mathrm{d}z_i$$

and

$$\mathrm{d}_z\kappa(\bar{z}\partial_z + z\partial_{\bar{z}}) = 0, \quad \mathrm{d}_z\kappa(i\bar{z}\partial_z - iz\partial_{\bar{z}})) = 0.$$

The function κ is proper on $\{|f| \leqslant R\}$ for any R so our claim follows. The existence of this function may seem miraculous, but as was explained to me by Maksim Maydanskiy, it can be nicely interpreted using the moment map of the $\mathrm{O}(n)$-action $g(x+iy) = gx + igy$ for $g \in \mathrm{O}(n)$, which preserves f (ask him !).

It is possible to explicitly compute the monodromy m around the loop se^{it}, $t \in [0, 2\pi]$ and get an exact symplectomorphism of $\mathrm{T}^*\,\mathrm{S}^{n-1}$ by conjugating with Φ_s. However it is not compactly supported so does not fit the framework introduced before: m is not an element of $\mathrm{Symp}^e(\mathrm{T}^*\,\mathrm{S}^n)$. We follow another approach which will help us to deform λ to get an actual Lefschetz fibration in the sense of definition 2.1. Consider the rays through the origin and their horizontal lifts. Using the fact that κ is horizontal and identifying $f^{-1}(0) \setminus \{0\}$ with $\mathrm{T}^*\,\mathrm{S}^{n-1} \setminus \mathrm{S}^{n-1}$ via Φ_0, this uniquely defines a diffeomorphism

$$\Psi : \mathrm{T}^*\,\mathrm{S}^{n-1} \setminus \mathrm{S}^{n-1} \times \mathbb{C} \to \mathbb{C}^n \setminus \kappa^{-1}(0)$$

which satisfies $f \circ \Psi = \mathrm{pr}_2$. To get an explicit formula, one computes the horizontal lift $\widetilde{\partial_r}$ of $\partial_r = \frac{x\partial_x + y\partial_y}{r}$:

$$\widetilde{\partial_r} = \frac{f(z)\bar{z}}{2|f(z)||z|^2}$$

A computation shows that $i_w^*(\widetilde{\partial_r}.\lambda)$ for all fibers $i_w : f^{-1}(w) \to \mathbb{C}^n$, and hence we know a priori the general expression of the Liouville form:

$$\Psi^*\lambda = \lambda_{\mathrm{can}} - K\mathrm{d}\theta$$

To compute an explicit expression for Ψ, we have to solve the differential equation

$$\dot{z}_t = \frac{e^{i\theta}\bar{z}_t}{2|z_t|^2}.$$

Since κ is constant along the trajectory, we have $|z_t|^4 - t^2 = |z_0|^4 = 4|p|^2$ (note that $\kappa \circ \Phi_0^{-1} = 4|p|^2$). Now a straightforward computation (see exercise 13) yields the following formula:

$$\Psi(p, q, re^{i\theta}) = \frac{|p|^{\frac{1}{2}}}{\sqrt{2}}(((1 + \frac{r^2}{4|p|^2})^{\frac{1}{2}} + 1)^{\frac{1}{2}}(q - i\frac{p}{|p|}) + ((1 + \frac{r^2}{4|p|^2})^{\frac{1}{2}} - 1)^{\frac{1}{2}} e^{i\theta}(q + i\frac{p}{|p|}))$$

From this expression we compute K:

$$K(p, q, re^{i\theta}) = K_r(p) = \frac{1}{2}\left(|p| - (|p|^2 + \frac{r^2}{4})^{\frac{1}{2}}\right).$$

The monodromy around the loop re^{it}, $t \in [0, 2\pi]$ is the time 2π-flow of K_r: indeed $X_{K_r} + \partial_\theta$ is the horizontal lift of ∂_θ for it spans the kernel of $\mathrm{d}\lambda$ on the cylinder of radius r. It is not clear however that this monodromy smoothly extends to the zero-section because $|p|$ is not smooth there but we claim this is the case. The hamiltonian K_r is the sum of terms and they Poisson commute because they are functions of $|p|$. The Hamiltonian $|p|$ generates the normalized geodesic flow:

$$\sigma_t(p, q) = (\cos(t)p - \sin(t)|p|q, \cos(t)q + \sin(t)\frac{p}{|p|}).$$

At time π (or 2π for $\frac{|p|}{2}$), this map is the differential of the antipodal map. which clearly extends to the zero-section. Hence the time 2π-flow of K_r is a well-defined exact symplectomorphism of $\mathrm{T}^* \mathrm{S}^n$. The last issue is that it is not compactly supported, so we truncate this hamiltonian: for $\rho > 0$, we pick a function $g(t)$ which equals 0 near 0, 1 near ρ and $g'(t) \geqslant 0$. We claim that the 1-form

$$\lambda' = \Psi_*(\lambda_{\mathrm{can}} + (g(|p|) - 1)K_r(p)\mathrm{d}\theta)$$

smoothly extends to $\mathbb{C}^n$ (see [Sei03]).

For this new form λ' the monodromy around a circle centered at 0 is an element of $\mathrm{Symp}^e(\mathrm{T}^* \mathrm{S}^{n-1})$, we call it a *symplectic Dehn twist.* Note that by chosing ρ very small, we can make the support of the symplectic Dehn twist contained in an arbitrarily neighborhood of the zero-section.

Vanishing cycles Let $\gamma : [0, 1] \to \mathbb{C}$ be an embedded path with $\gamma(1) = 0$. As we shall see, associated to this path is a Lagrangian disk in $\mathbb{C}^n$ called the *Lefschetz thimble*, with boundary an exact Lagrangian sphere in $f^{-1}(\gamma(0))$ called the *vanishing cycle.* Let $\widetilde{\gamma} : [0, 1[\to \mathbb{C}^n$ be an horizontal lift, this is well-defined because the function

κ is preserved. Then $\widetilde{\gamma}$ converges to 0 when $t \to 1$ if and only if $\kappa = 0$ along the path. Hence the union of all the points that parallel transport to 0 above the path γ is precisely $D_\gamma = f^{-1}(\gamma) \cap \kappa^{-1}(0)$. We claim that D_γ is a smooth Lagrangian disk which intersect the fiber over $\gamma(t)$ in an exact Lagrangian sphere for $t < 1$ and a single point (the origin) for $t = 1$. Let us first look at the case where $\gamma(t) = 1 - t$. Then the equations for D_γ are $\langle x, y\rangle = 0, 0 \leqslant |x|^2 - |y|^2 \leqslant 1, |x||y| = 0$ or equivalently $y = 0, |x| \leqslant 1$, which is obviously a smooth Lagrangian disk, λ actually vanishes on D_γ. For the general case we make two observations. First, for z in the fiber over $se^{i\theta}$, one notices that $ze^{-i\theta/2}$ lies in the fiber over s, and hence $\kappa^{-1}(0) \cap f^{-1}(se^{i\theta}) = e^{-i\frac{\theta}{2}} L_s$ where $L_s = \{y = 0, |x| = \sqrt{s}\} = \Phi_s^{-1}(\mathrm{S}^{n-1})$. This is an exact Lagrangian sphere because the unitary map $z \mapsto e^{-i\frac{\theta}{2}} z$ preserves λ. Moreover, in the neighborhood of 0, D_γ can be mapped to the subspace $\{y = 0\}$ via the diffeomorphism $z \mapsto e^{-i\theta(z)/2} z$ where $\theta(z)$ is smooth determination of the argument of $f(z)$ defined on D_γ near 0. This shows that D_γ is a smooth embedded disk, the fact that it is Lagrangian can be deduced from the following: it intersects the regular fibers in Lagrangian submanifolds and it is horizontal.

Here is another argument which is less elementary for it involves the stable manifold theorem but applies more generally, especially in the case where ω is not assumed to be standard near the critical points. This trick is attributed to Donaldson in [Sei03]. Pick a function h defined in a neighborhood V of the image of γ, which vanishes transversely along γ and such that $(\gamma'(t), \nabla h(\gamma(t)))$ is an oriented $\mathbb{R}$-basis of $\mathbb{C}$, and $H = h \circ f$ a Hamiltonian defined near $f^{-1}(V)$. We claim that X_H has an hyperbolic zero at the origin and that its trajectories over γ are horizontal and converge to 0. Then D_γ can be alternatively defined as the stable manifold of the origin for X_H and it follows from the stable manifold theorem that D_γ is a smooth embedded disk. The fact that D_γ is Lagrangian can be deduced from the proof of proposition 1.4.

When we vary the path by a homotopy, the corresponding vanishing cycles moves by an exact Lagrangian isotopy, which can then be extended as a Hamiltonian isotopy of the fiber. Also, it is possible to deform a vanishing cycle by a Hamiltonian isotopy by deforming the Lefschetz fibration.

Matching cycles Let γ be a path joining two critical values. In the middle of the path, there are two vanishing cycles coming from both sides, if they are Hamiltonian isotopic, then after a suitable deformation the Lefschetz fibration, the Lefschetz thimbles glue into a Lagrangian sphere in the total space. It is an interesting way of producing Lagrangian spheres in the total space.

2.5 Examples

The trivial fibration Let (F, λ_F) be a complete finite type Liouville manifold, $W = F \times \mathbb{C}$ endowed with $\lambda = \lambda_F + \frac{1}{2}r^2 \mathrm{d}\theta$ and $f = \mathrm{pr}_2 : W \to \mathbb{C}$. The contact boundary M has an open book decomposition with page F and monodromy id. When $F = \mathbb{C}^{n-1}$, we find the standard open book decomposition of S^{2n-1} with page $\mathbb{C}^n$ and monodromy id. The unit sphere actually represents an adapted contact form in this case (exercise 11).

One critical point Consider the map $f : \mathbb{C}^n \to \mathbb{C}$ given by $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$, and equip $\mathbb{C}^n$ with the Liouville structure λ' constructed in the previous section. Using Thurston's trick plus some tweaking near the vertical boundary as explained in a previous section, we can make $\mathbb{C}^n$ into a complete Liouville manifold with compatible Lefschetz fibration $f' : \mathbb{C}^n \to \mathbb{C}$ which has just one critical point, fiber $\mathrm{T}^* \mathrm{S}^{n-1}$, vanishing cycle the zero-section and monodromy a symplectic Dhen twist about the zero-section. In particular it exhibits the standard sphere S^{2n-1} as an open book with page $\mathrm{T}^* \mathrm{S}^{n-1}$ and monodromy a symplectic Dehn twist (see exercise 12).

Two critical points Consider the affine quadric $W = \{z_1^2 + \dots + z_n^2 + w^2 = 1\}$ in $\mathbb{C}^{n+1}$ and the map $f : W \to \mathbb{C}$ equal to the coordinate w. There are exactly two critical points with critical values 1 and -1. The fiber is $\mathrm{T}^* \mathrm{S}^{n-1}$ and the vanishing cycles are both equal to the zero-section. The total space is $\mathrm{T}^* \mathrm{S}^n$.

Milnor fiber Consider $W = \{z_1^2 + \dots + z_n^2 + w^{k+1} = 1\}$ with the projection to w. The critical values are the $(k+1)$-th roots of unity. The total space is called the A_k Milnor fiber, and can be described as a plumbing of k copies of $\mathrm{T}^* \mathrm{S}^n$. The k zero sections of these $\mathrm{T}^* \mathrm{S}^n$ form a chain (A_k-type) of Lagrangian spheres and can be described as matching cycles corresponding to all the sides of the $k+1$-gon (with vertices the roots of unity) but one.

2.6 Basis of vanishing cycles, Hurwitz moves

Let $W \to \mathbb{C}$ be a Lefschetz fibration assume 0 is a regular value and denote by F the fiber over 0. Pick a *basis of vanishing paths*: $\gamma_1, \dots, \gamma_m$ disjoint embedded paths from 0 to the critical points, ordered consistently with the natural counter-clockwise cyclic order in a neighbourhood of zero. This gives a collection $(L_1, \dots, L_m)$ of exact

Lagrangian spheres (the vanishing cycles) in F. This collection itself is not an invariant of the Lefschetz fibration because it depends on the choice of vanishing paths, however two such choices differ by a finite number of the following modification:

- *isotopies*
- *cyclic permutation* of $(\gamma_1, \dots, \gamma_m)$.
- *Hurwitz moves*: for $i \in \mathbb{Z}/m$, change γ_i, γ_{i+1} for γ'_i, γ'_{i+1} with either $\gamma'_i = \gamma_{i+1}$ and $\gamma'_{i+1} = \beta_{i+1}^{-1}\gamma_i$, or $\gamma'_i = \beta_i \gamma_{i+1}$ and $\gamma'_{i+1} = \gamma_i$, where β_i is a small loop bases at 0 encircling p_i and very close to the path γ_i. The new vanishing cycles are $(L'_i, L'_{i+1}) = (L_{i+1}, \tau_{L_{i+1}}^{-1}(L_i))$ or $(\tau_{L_i}(L_{i+1}), L_i)$, the other ones being unchanged.

One can interpret this in terms of the Braid group on m strands B_m and the Hurwitz moves correspond to the standard generator σ_i and its inverse.

Conversely, this can be used to construct Liouville manifolds: given a Liouville manifold F with a sequence $(L_1, \dots, L_m)$ of parametrized exact Lagrangian spheres, it is possible to construct a Liouville manifold W with a Lefschetz fibration $W \to \mathbb{C}$ with critical values ζ^i, for $i = 0 \dots m-1$ and $\zeta = e^{\frac{2i\pi}{m}}$, and vanishing cycles associated to the rays from the origin are precisely $L_1, \dots, L_m$. Changing $(L_1, \dots, L_m)$ by Hamiltonian isotopies in F, cyclic permutations or Hurwitz moves does not change the total space because it just corresponds to another choice of basis of vanishing paths.

There is yet another operation which leaves the total space W unchanged:

- *stabilization*: attach a Weinstein handle to F along a Legendrian sphere which bounds an exact Lagrangian disk L and replace $(L_1, \dots, L_m)$ by $(L_1, \dots, L_m, L_{m+1})$ where L_{m+1} is the union of L and the core of the Weinstein handle.

This can been interpreted as composing W with a Weinstein cobordism with a pair of critical points of index $n-1$ and n in cancellation position.

2.7 Weinstein structures and Lefschetz fibrations

If $f : W \to \mathbb{C}$ is a Lefschetz fibration with fiber F which is itself a finite type Weinstein manifold, then W is also Weinstein. We start with an exhausting convex function ϕ_0 on the central fiber F_0. By Thurston's trick as explained above, one can make the Liouville vector field transverse to small tubes $\{|f| = r\}$ around the central fiber. Hence $\phi = \phi_0 + |f|^2$ is a Lyapunov function for X near F_0. As we let the radius of this tube grow, it will meet critical points of f. When this occurs, this corresponds to the attachment of a critical Weinstein handle along the corresponding vanishing cycle (lifted as a Legendrian in the boundary of the tube). The Lyapunov function ϕ can be extended over the handle.

The converse statement is much harder and makes use of Donaldson's techniques.

Theorem 2.6 (Giroux-Pardon [GP14]). *Any Weinstein manifold admits a Lefschetz fibration over $\mathbb{C}$ with Weinstein fiber.*

2.8 Exercises

Exercise 11. *Show that on S^{2n-1}, the standard contact structure $\xi = \ker(\sum r_i^2 \mathrm{d}\theta_i)$ is carried by the open book $(B = \{r_n = 0\}, \theta = \theta_n)$. Hint: the contact form $\alpha = \frac{1}{2}\sum_{i=1}^n r_i^2 \mathrm{d}\theta_i$ is adapted.*

Exercise 12. *For $n = 2$, instead of $f = z_1^2 + z_2^2$ we consider $f = z_1 z_2$. Prove that the unit sphere is an adapted contact form for the open book $\theta = \theta_1 + \theta_2 = \arg(f)$.*

Exercise 13. *Check the computations of the radial trivialization for $f = z_1^2 + \cdots + z_n^2$.*

Exercise 14. *In the construction of the contact manifold associated to a symplectic open book, show how to exhibit an "arbitrary large" neighborhood of the binding: for any contact form α for B and $R > 0$, there exists an embedding $i : B \times \mathrm{D}_R^2 \to \overline{M_\phi}$ which is the identity on $B \times \{0\}$ and satisfies $i^*\xi = \ker(\alpha + r^2 \mathrm{d}\theta)$.*

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