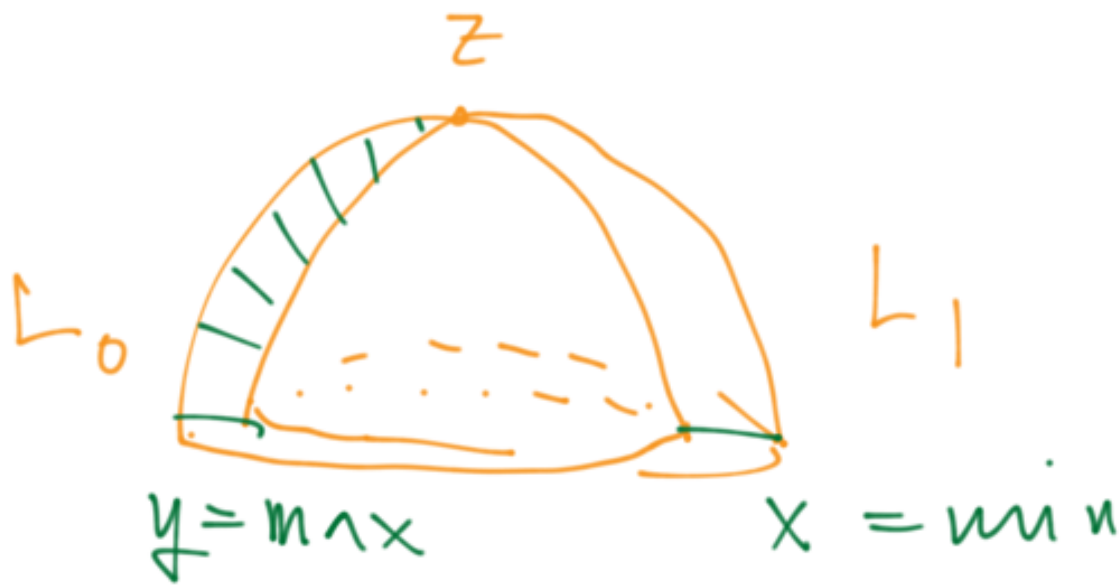
A surgery map for SH



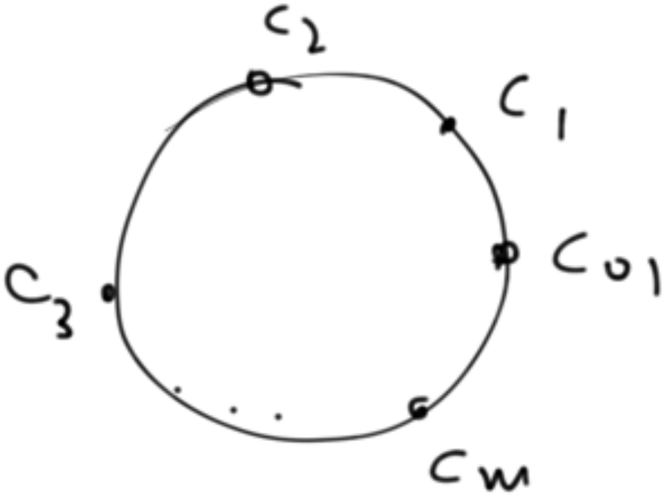
$A^{\text{Ho}}(\Lambda)$ generated by

$c_0, c_1 \dots c_m$

$c_0 \in Ch(\Lambda_0, \Lambda_1) \cup z$

$c_j \in Ch(\Lambda)$

We think of such words as written on the circle with c_{01} distinguished and identify words by cyclic perturbation.



$$|c_{01}| = |c| + 1$$

$$|x| = 0; \; |y| = n-1;$$

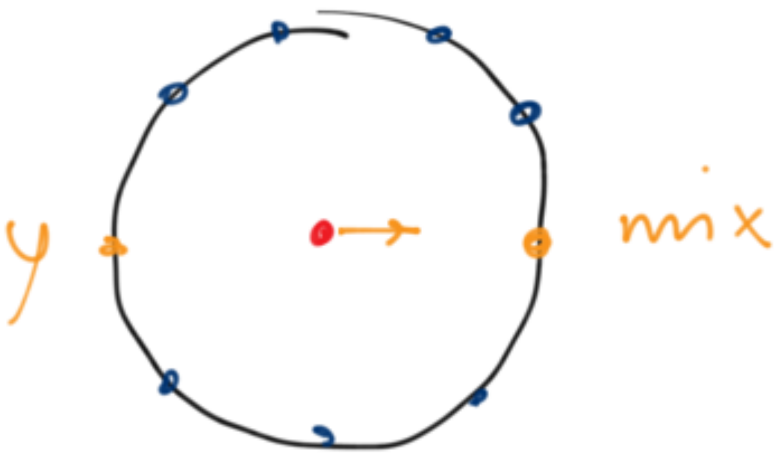
$$|z| = n.$$

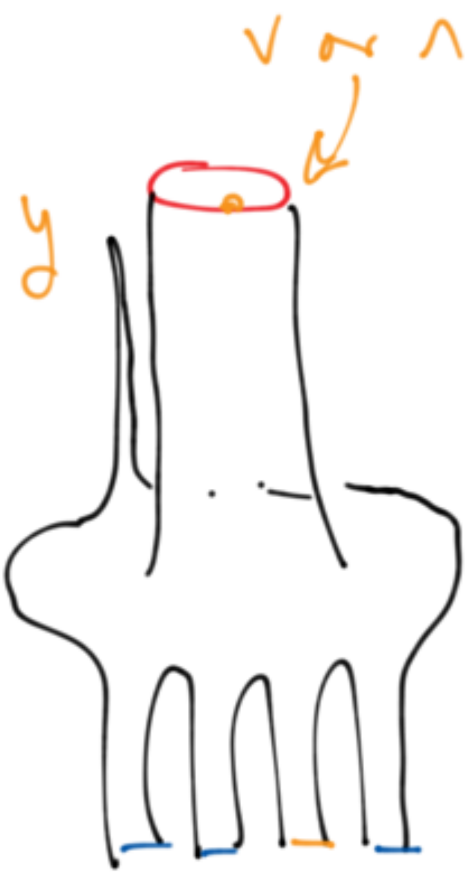
$$c_{01} c_1 \cdots c_m \sim (-1)^{|c_m| \cdot (|c_1 \cdots c_{m-1}|)} \, c_m c_{01} c_1 \cdots c_{m-1}$$

We define a differential on $C(X_0) \oplus A^{H_0}(\Lambda)$:

$$d = \begin{bmatrix} d_{SH} & 0 \\ \delta & d_A \end{bmatrix}$$

Here δ counts.



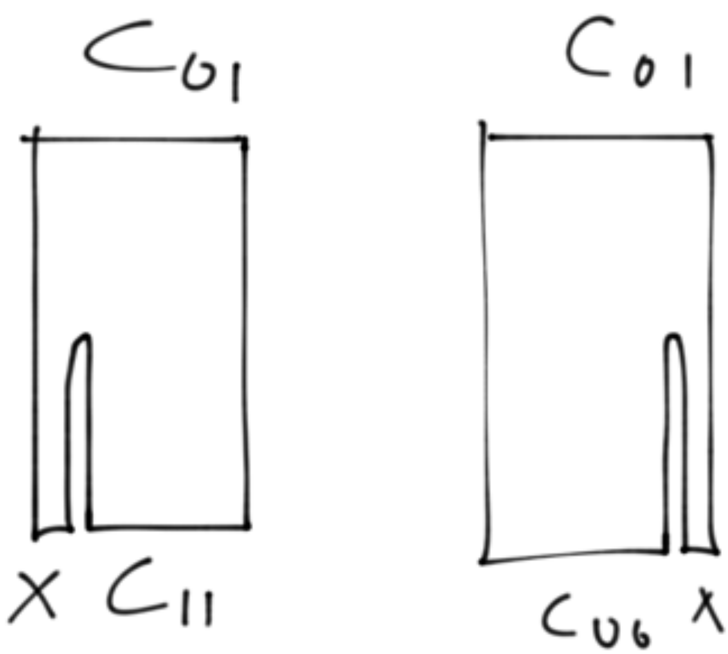


<u>Note</u>

$$Ch(\Lambda_1) = Ch(\Lambda_0) = Ch(\Lambda)$$

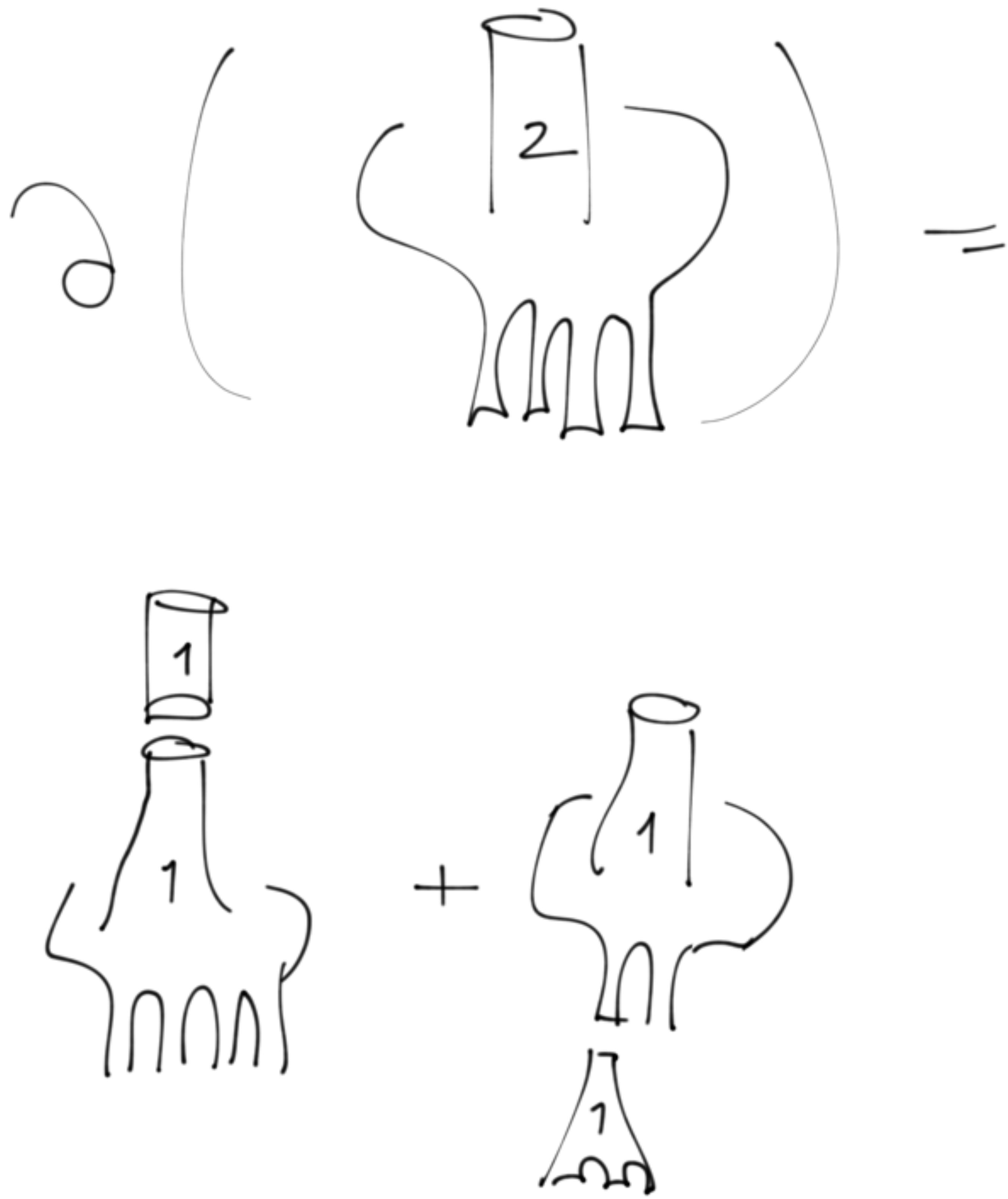
In d_A : $d_A z = y$ and

c — chord of Λ



$$dc_{01} = cx - xc + \dots$$

Looking at the boundary of 1-dim moduli spaces we find that $d^2 = 0$.



There is a natural surgery chain map:

$$\Phi : SH(X) \longrightarrow SH(X_0) \oplus A^{H_0}(\Lambda)$$

which is a chain isomorphism.

Proof. The map Φ is defined like the differential

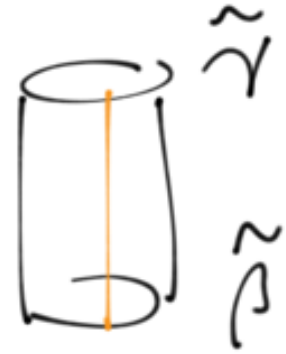
$$\Phi = \begin{array}{cc} \begin{bmatrix} \Phi_X & \varphi \end{bmatrix} \\ \begin{array}{cc} SC(X_0) & A^{H_0}(\Lambda) \end{array} \end{array}$$

Note $M_0(X) = M_0(X_0) \oplus \mathbb{Q}\cdot\tau$

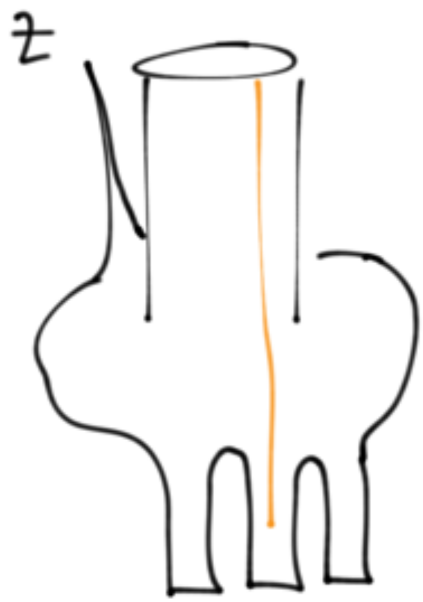
$\Phi_X|_{M_0(X_0)} = id$

$\Phi_X(\tau) = 0$; $\varphi(\tau) = x$

$\Phi_X(\tilde{\gamma})$ counts

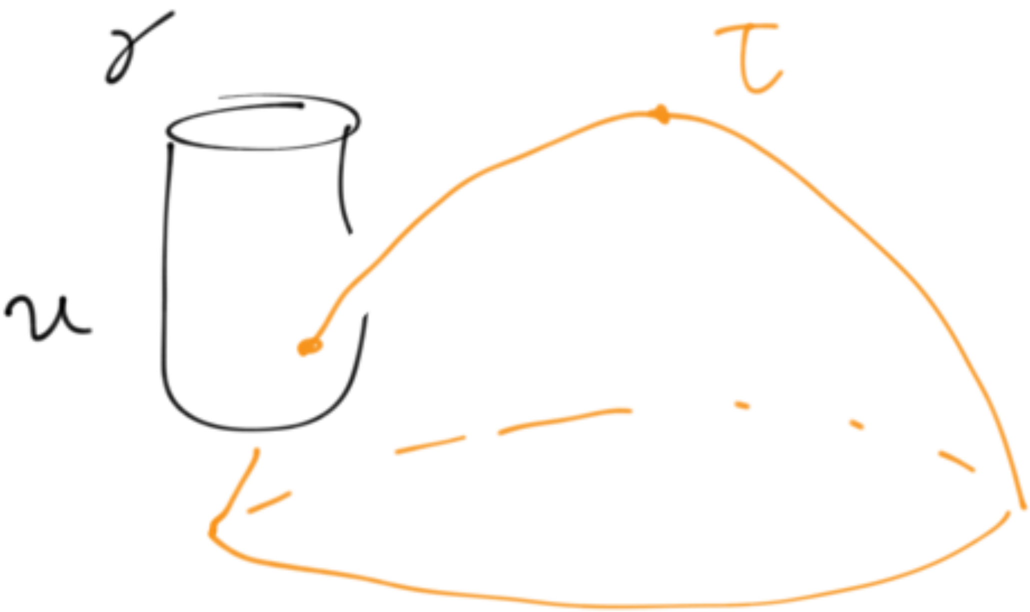


$\varphi(\tilde{\gamma})$ counts



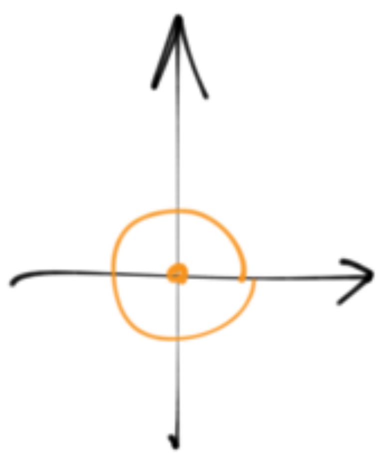
The chain map eqn is proved as before with one new ingredient.

$$\check{\gamma} \xrightarrow{d} \tau \xrightarrow{\Phi} X$$



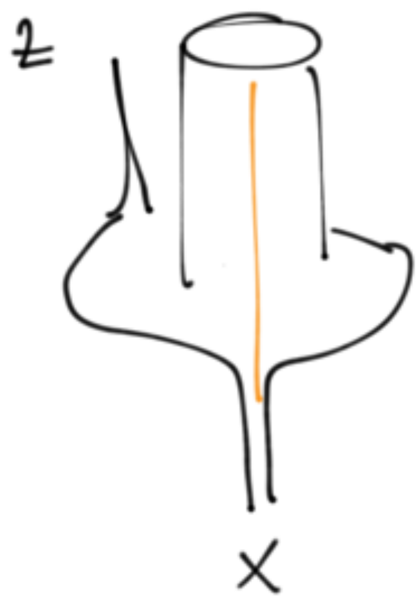
u is transverse to L and can be opened up

(or equivalently a const disk in L can be glued uniquely)

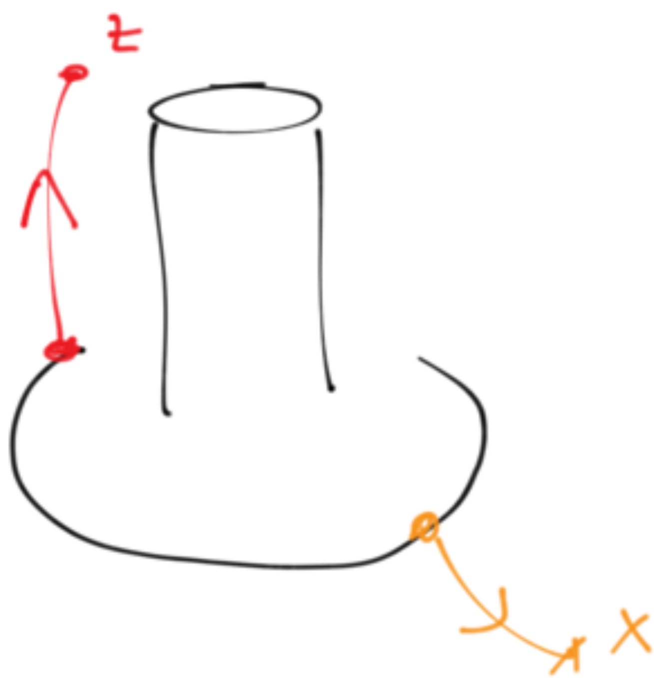


$x^2+y^2=\varepsilon \qquad \varepsilon \in [0,1]$

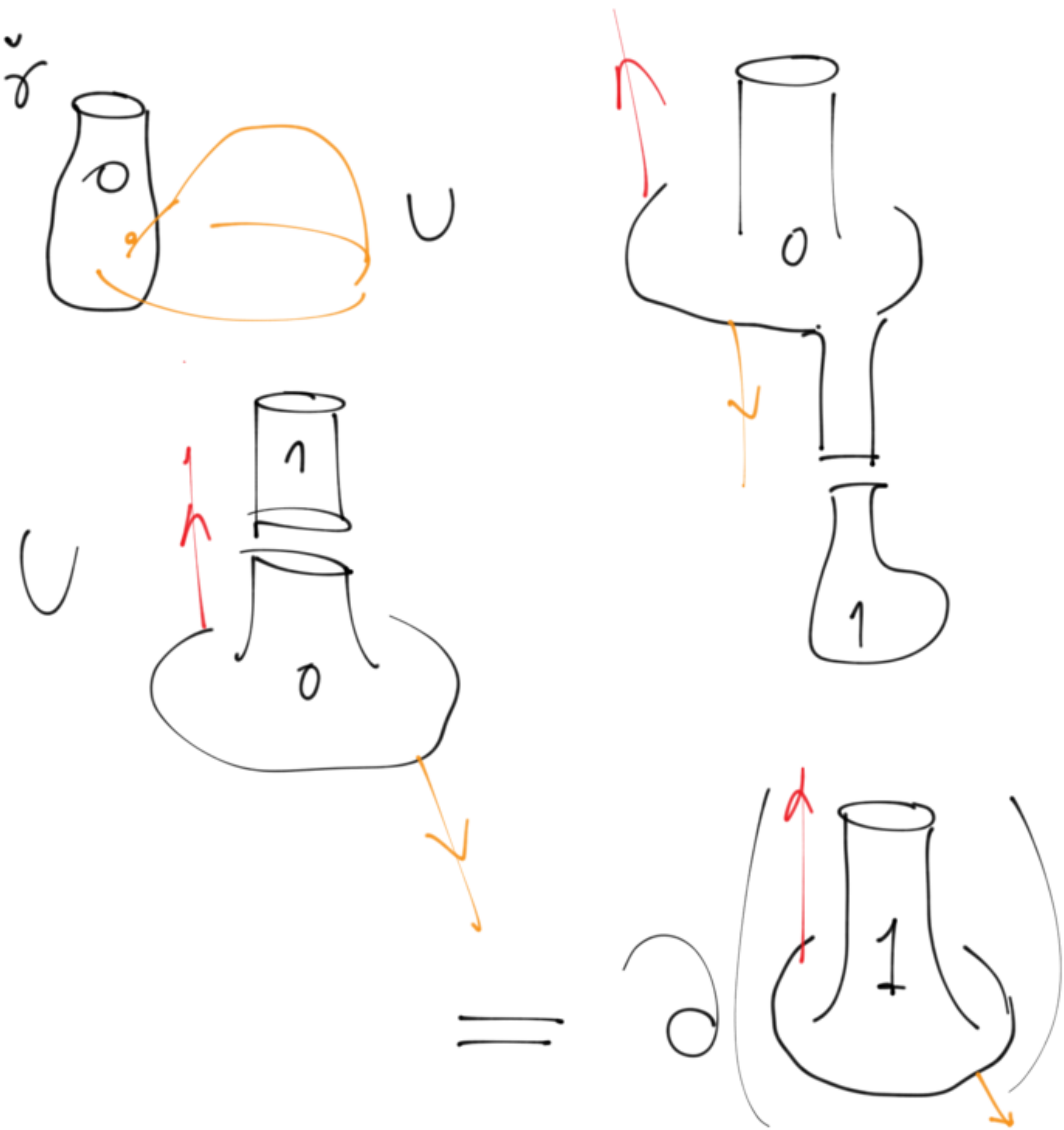
For sufficiently small separation between L_0, L_1



an in 1-1 corr with



S_0

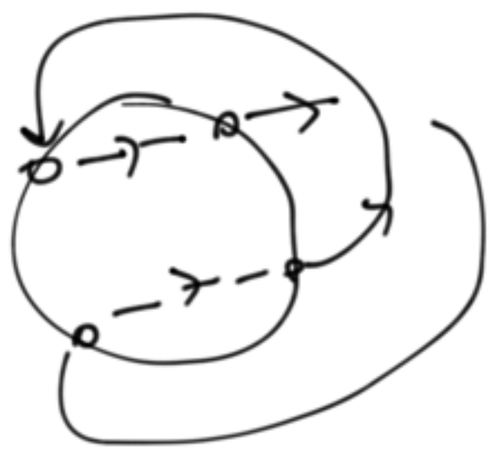


We next argue as for HW.

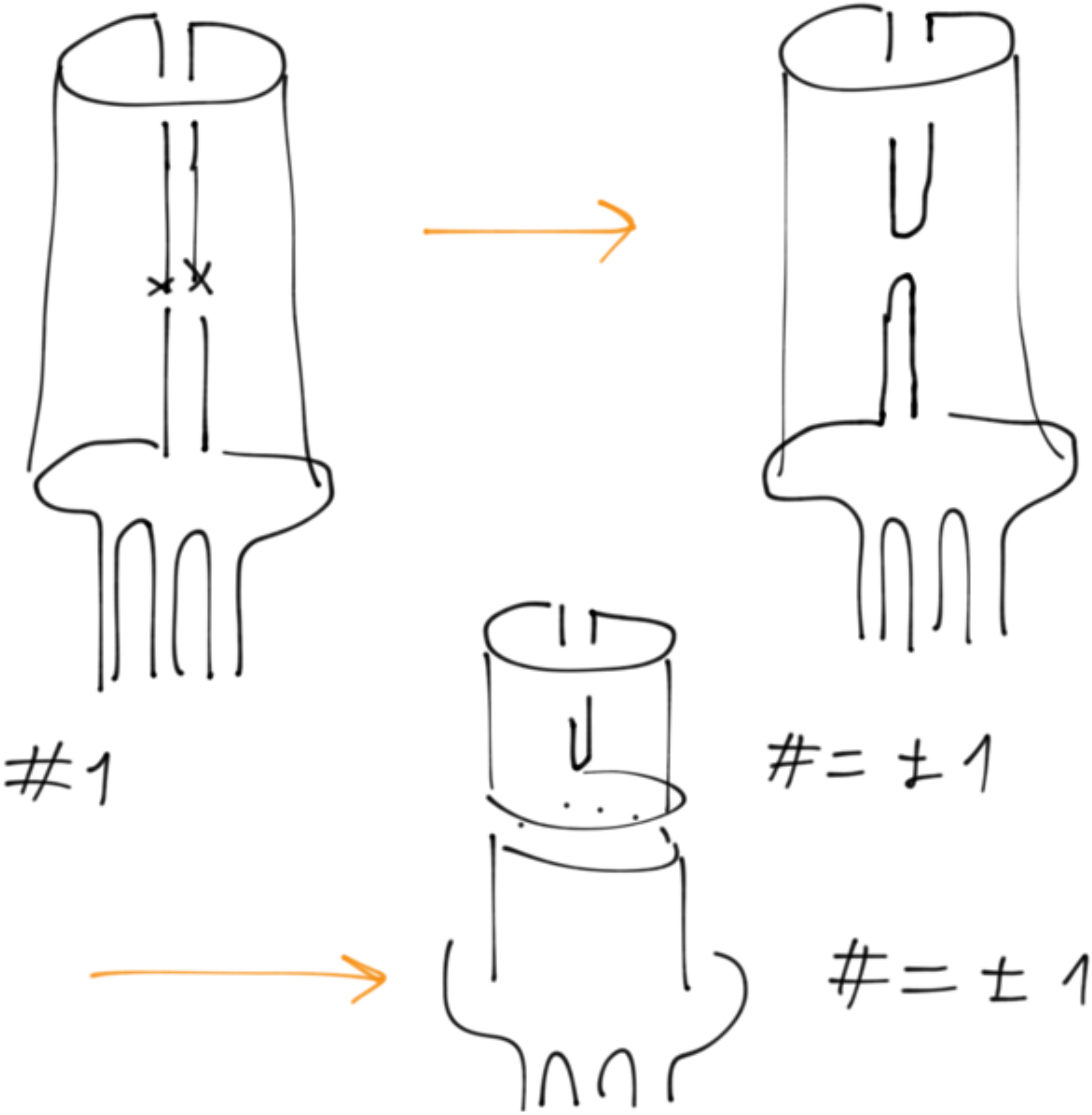
1) identify generators

2) Find 1's on diagonal (after taking the Morse differential).

The 1-1 correspondence between orbits and cyclic words is as for chords but uses iteration of flow maps



The basic curves are constructed as before by self-gluing



There is a orientation problem for bad orbits but the chain map respects action filtr.

On cyclic words:

$b_1 b_2 \ldots b_m$ is $\underline{\text{bad}}$ if even

b odd over of odd.

$$d_M(\underline{b}b) = xbb - bxb =$$
$$= xbb + xbb$$

So the induced map of spectral sequences is an iso from 2nd page. $\square$.

Conclusion.

$$SH(X) \simeq SH(X_0) \oplus LH^{Ho}(\Lambda)$$

$LH^{Ho}(\Lambda)$ subcomplex

$SH(X_0) \simeq 0$ if

X_0 subcritical and

then

$$SH(X) \simeq LH^{Ho}(\Lambda)$$

Example

Consider S^3 with one 1-handle $= X_0$

X contractible 4-mfd

$(\pi_1(\partial X) \neq 1)$

$LH^{\text{ho}}(\Lambda) \neq 0 \Rightarrow SH(X) \neq 0$

$X \times X \underset{\text{top}}{\approx} \mathbb{R}^8$ but $SH(X \times X) \neq 0$.

Example.

$SH(T^* S^n)$

$\hat{a} = a_{01}$

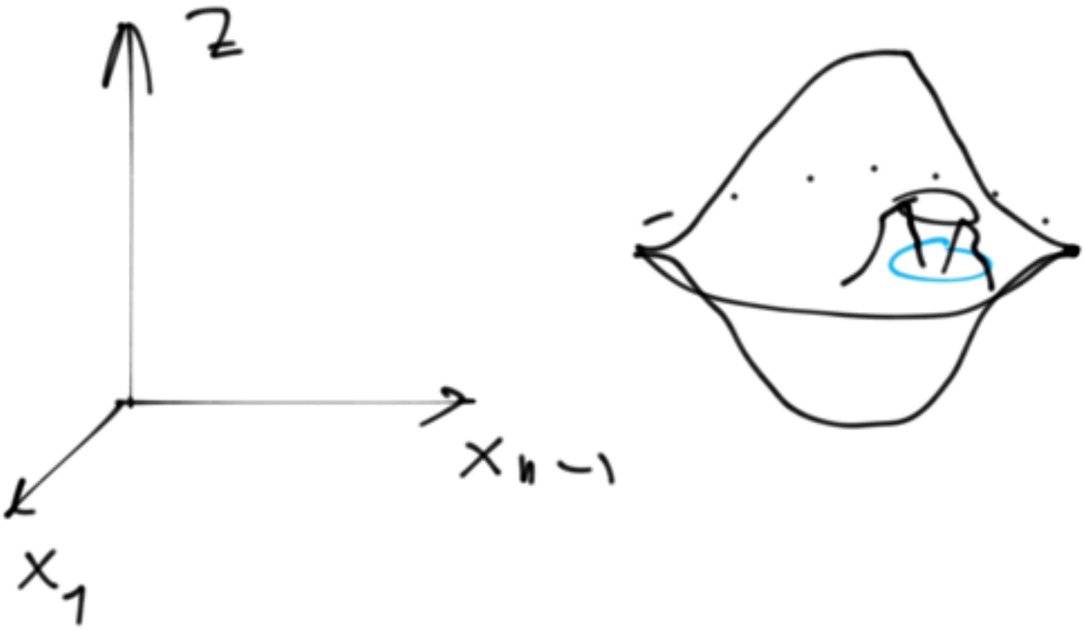
0	x
$n-1$	xa
n	$\hat{a}$
$2(n-1)$	xa^2
$2(n-1)+1$	$\hat{a}a$
$3(n-1)$	xa^3
$3(n-1)+1$	$\hat{a}a^2$

n – odd no diff'l

n – even diff'l

Example.

A fake T^*S^n ; X



$\rightsquigarrow$ loose Λ in same reg hom class

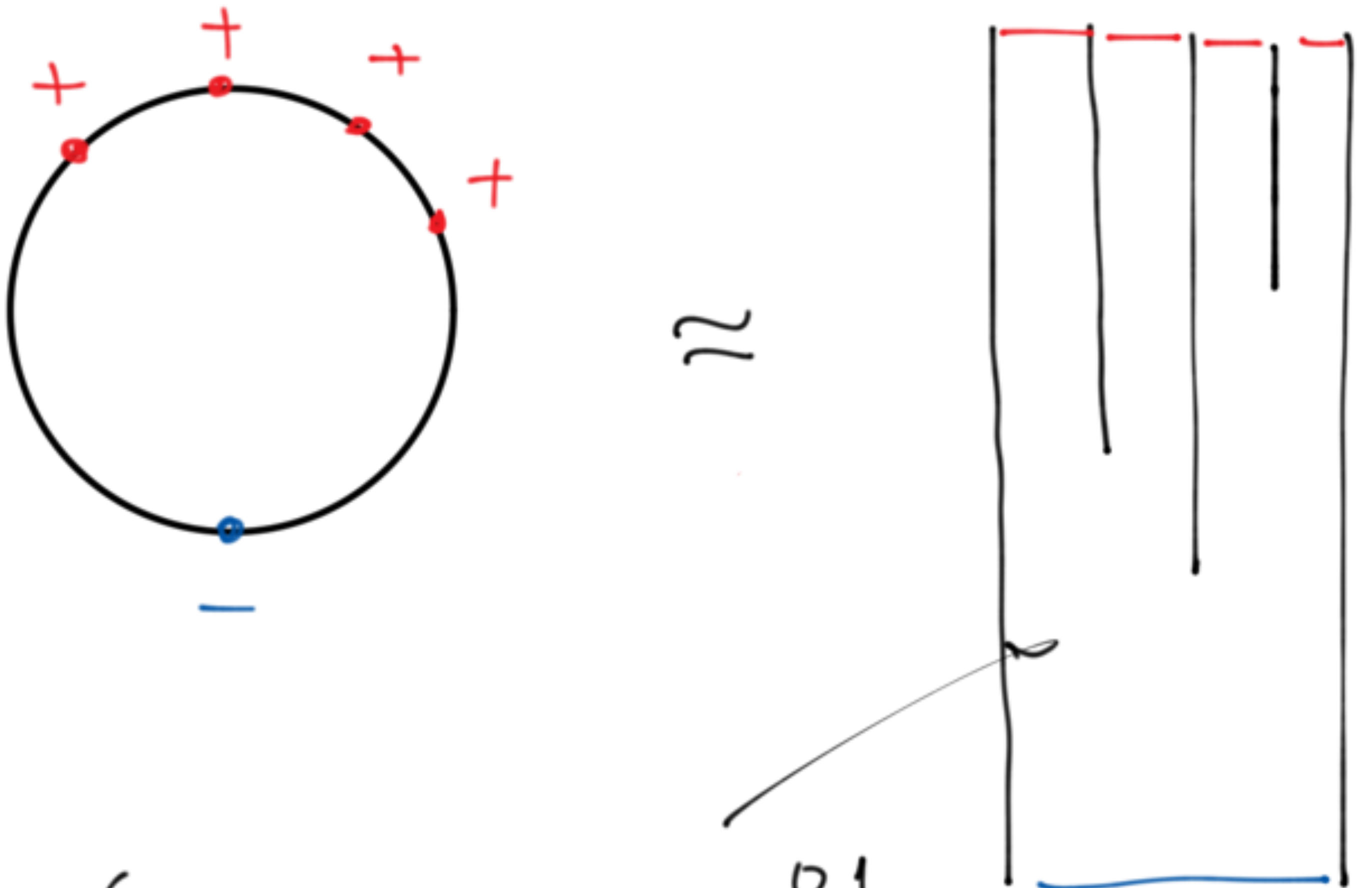
$\partial b = 1$

$\Rightarrow A^{th}(\Lambda)$ acyclic

$\Rightarrow SH(X) = 0.$

An A_∞ - isomorphism.

The wrapped Floer homology has a further multiplicative structure defined as follows in terms of the original WH-complex.



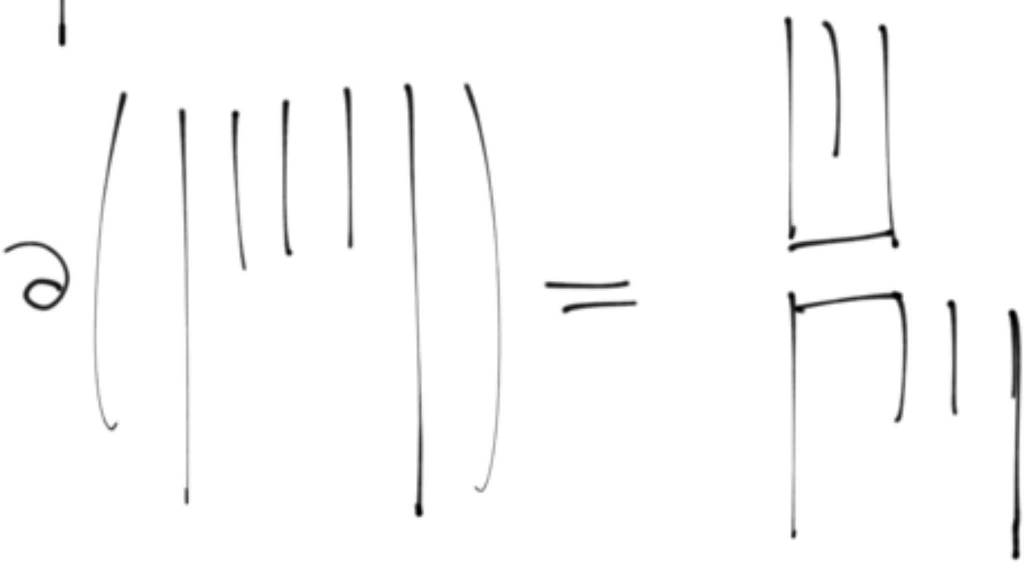
$$(u + X_H \otimes dt)^{0,1} = 0$$

This gives operations

$$\mu_k : \underbrace{CW \otimes \cdots \otimes CW}_{k} \longrightarrow CW$$

$$\mu_1 = \partial .$$

Looking at 1-dim mdli spaces



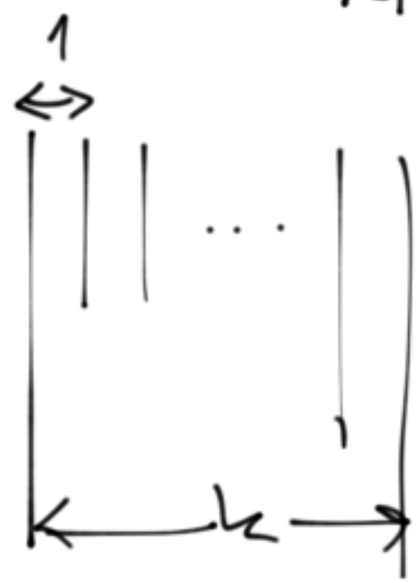
we find that the A_∞-relations, or

$$\mu \circ \mu = 0$$

holds.

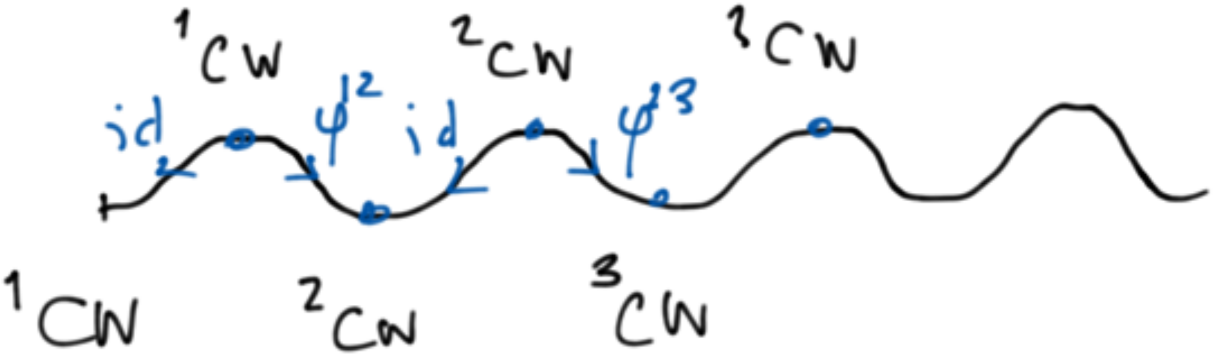
Techical problem

$$\underbrace{CW_H(L) \otimes \cdots \otimes CW_H(L)}_{k} \to CW_{kH}(L)$$



Replace CW with "mapping telescope"

$$CW \times \mathcal{M}orse([0,\infty)) = \widetilde{CW}$$



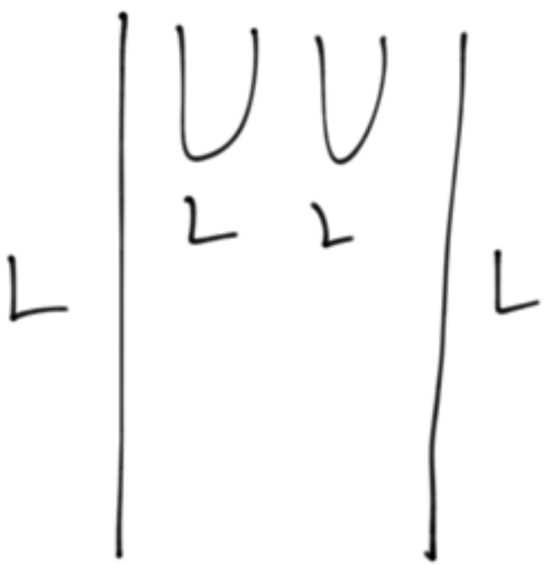
$\varphi^{j\,j+1}$ quasi-iso

$$\Rightarrow \quad \widetilde{CW} \simeq CW .$$

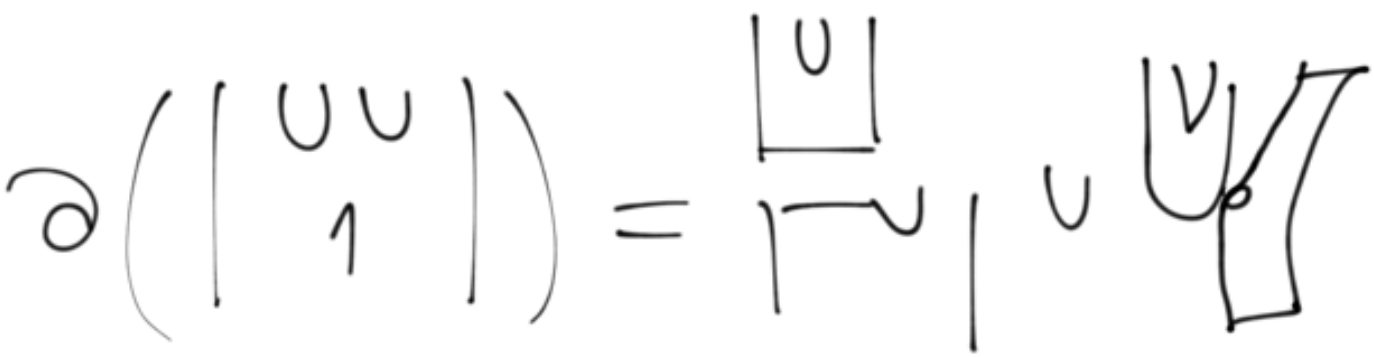
Now we can use the curves drawn.

To connect to surgery we need a purely holomorphic version of this structure. To see it we extend the additive isomorphism to an A_∞ -morphim.

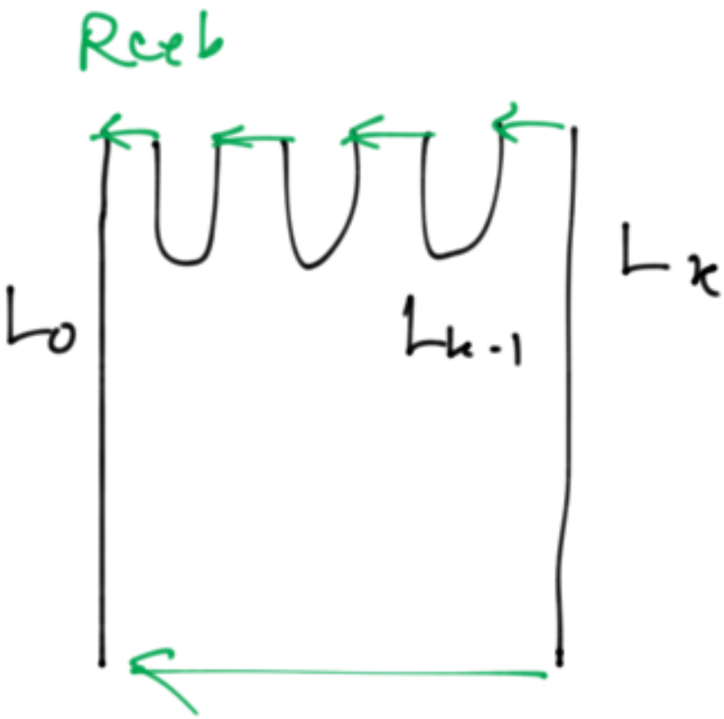
First attempt: μ_k defined by counting



Problem: boundary breaking.

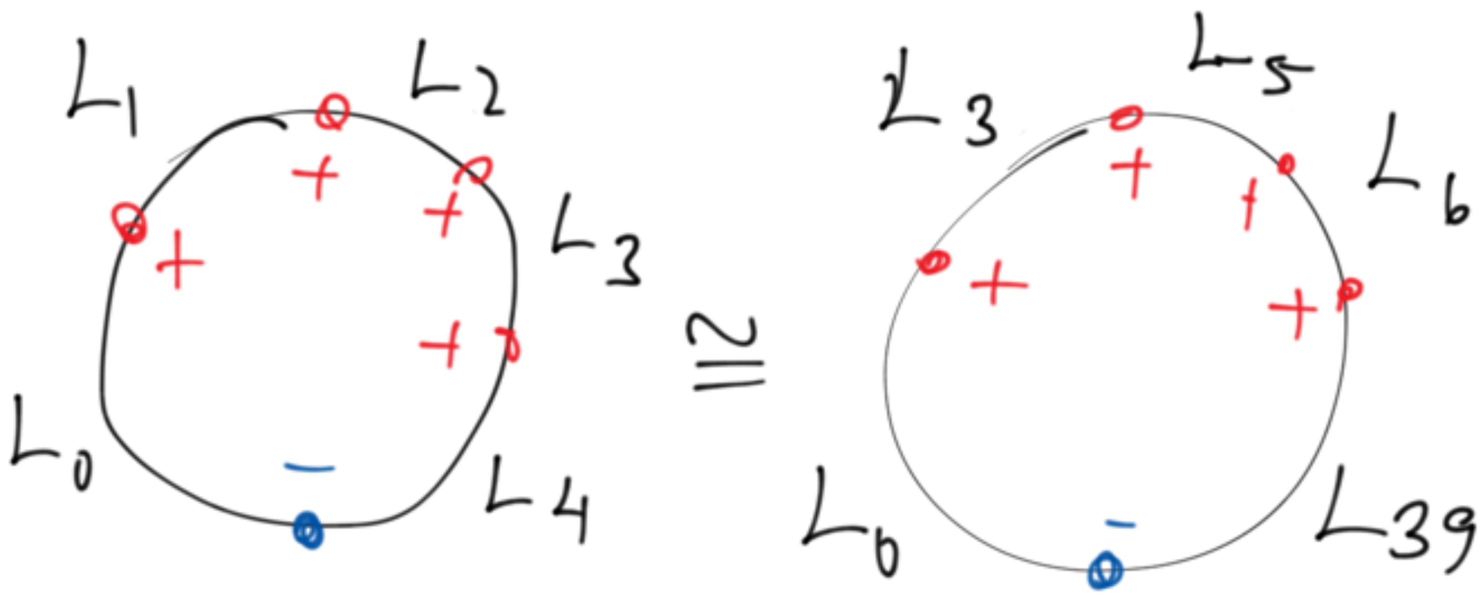


To overcome this we use parallel copies. For μ_k we use k nearby copies

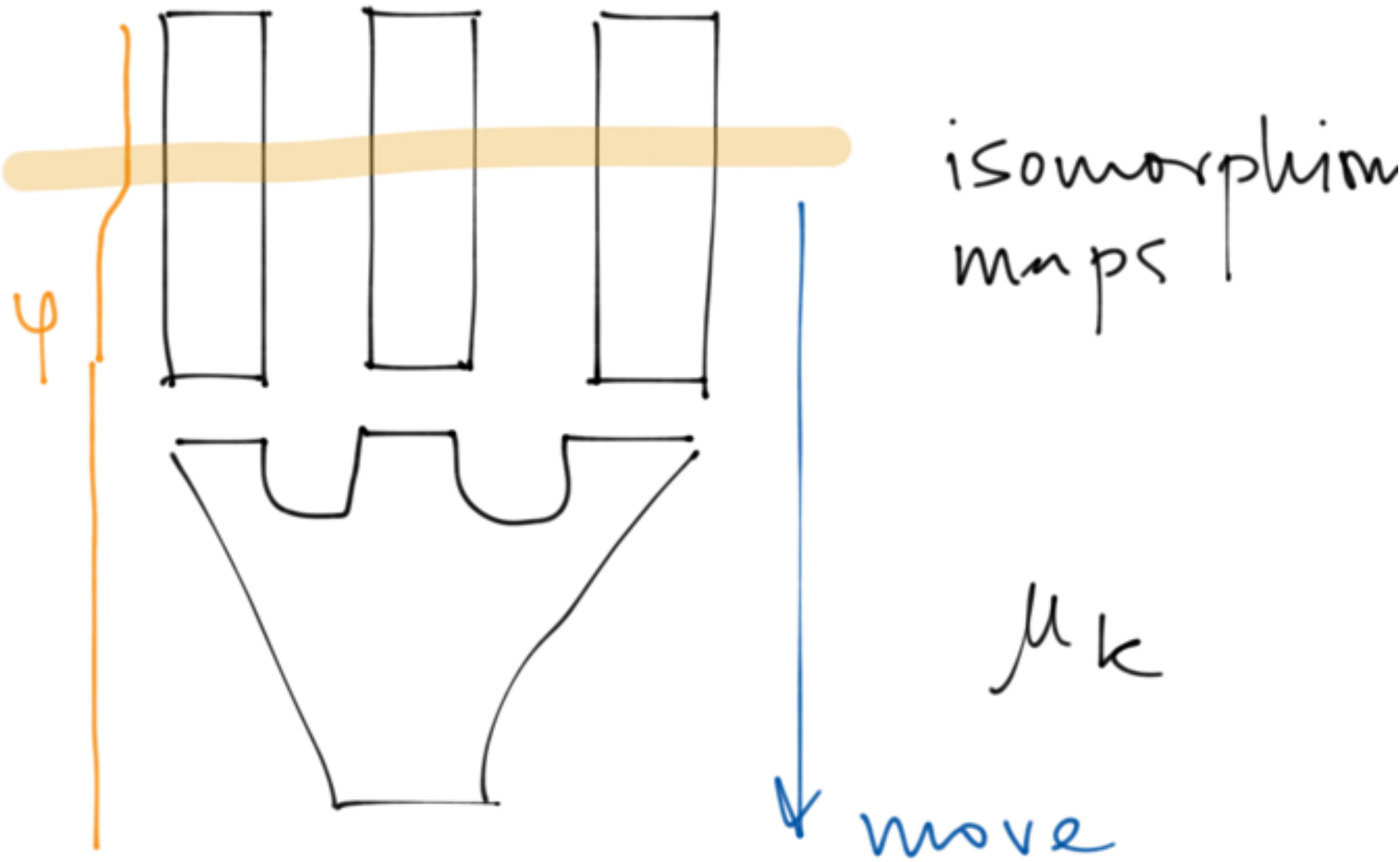


Now boundary splitting cannot happen for topological reasons.

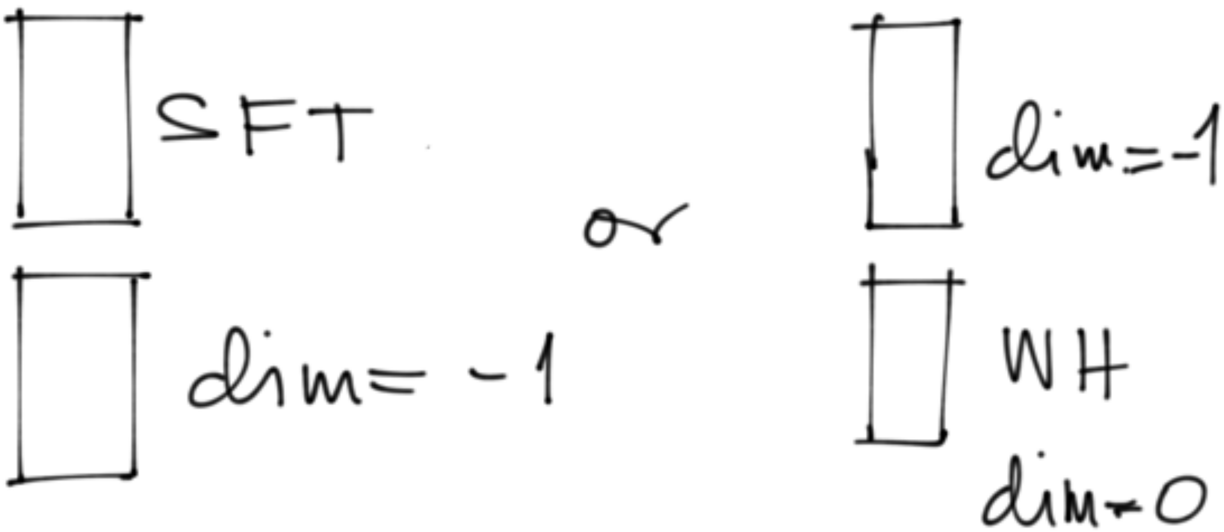
For sufficiently small shifts curves with boundary conditions on different parallel Lagrangians can be identified (by transversality)



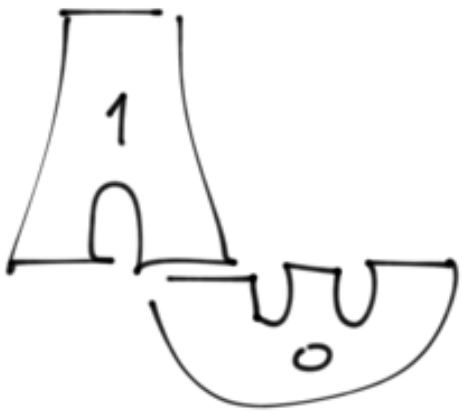
To find out which A_∞-structure to use:



As the region where φ' is supported moves we find splittings



Here the SFT part are holomorphic buildings of dim = 1



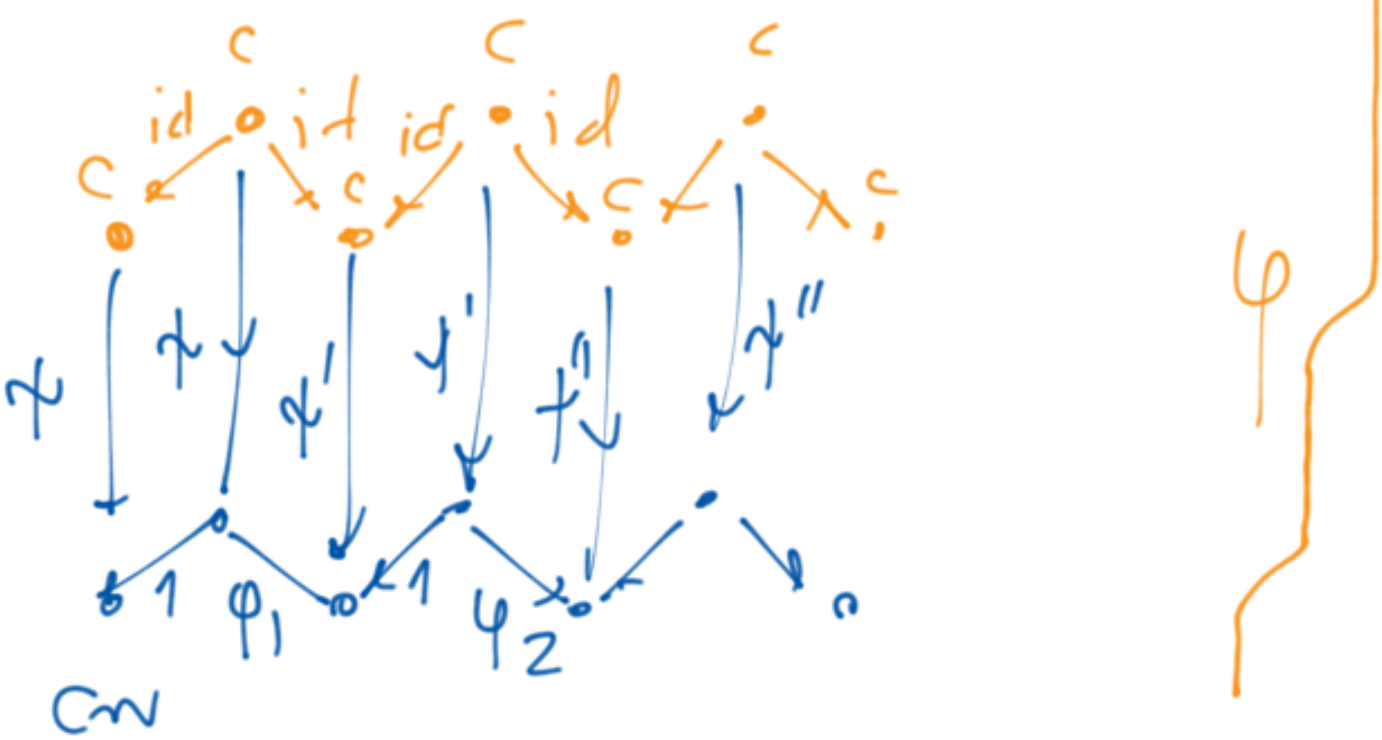
the (−1) curves give an A_∞-map

and the WH curves are μ_K.

We thus get an A_∞-map which is an isomorphism on E_1-page and hence iso.

More formally we apply this to telescopes (in fact removing them)

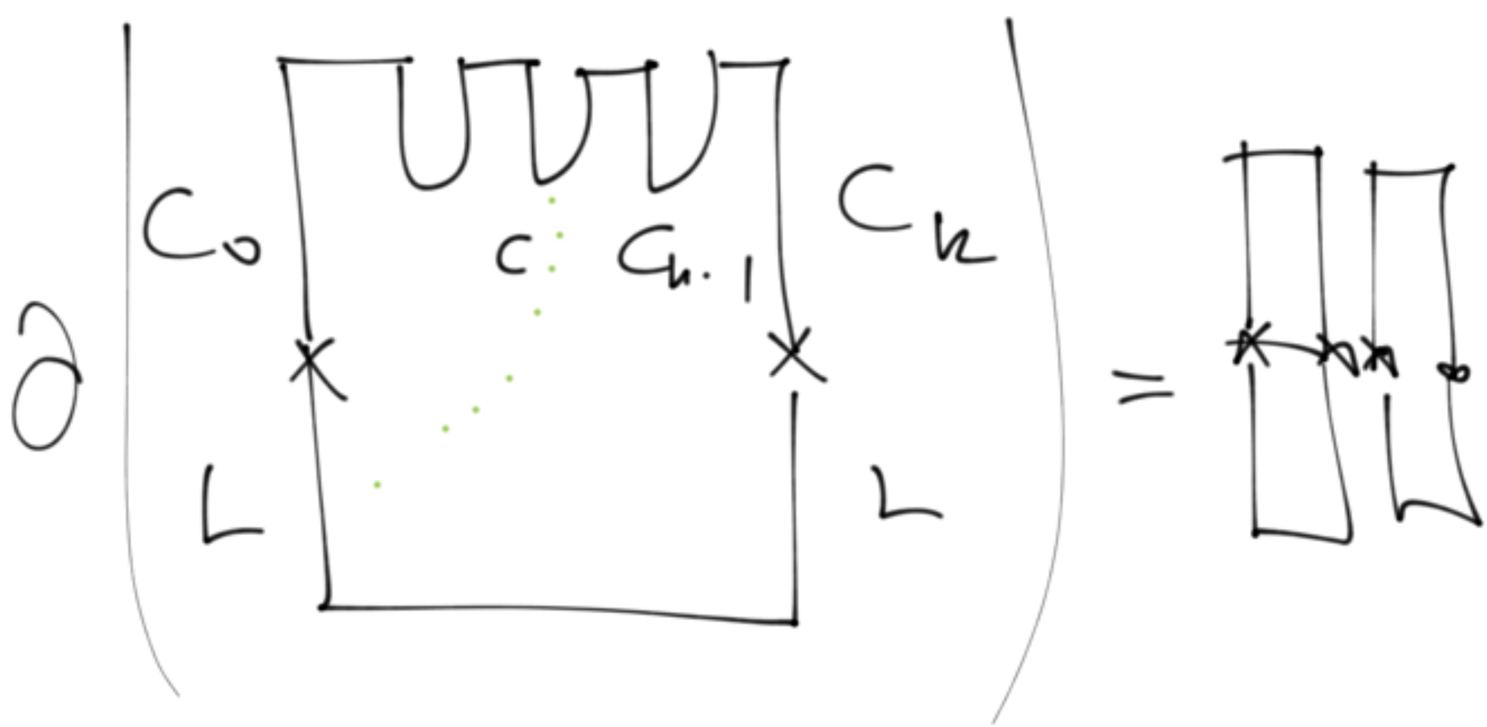
Let C denote the complex w/o Hamilt.



$$\psi^{(k)} = \varphi_k \psi^{(k-1)}$$

Similarly we extend the surgery map, where we use the very simple A_∞ structure on $A(\Lambda)$ which has μ_2 given by its multiplication and higher μ equal to 0.

The map counts



$\square$.

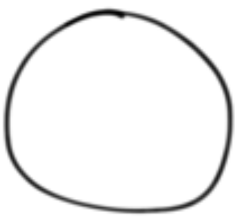
Conclusion.

There is an A_∞ q.isomorph

$$HW(C) \longrightarrow A(\Lambda)$$

Example.

$\overline{M}$ closed mfd, $M = \overline{M}$ - ball

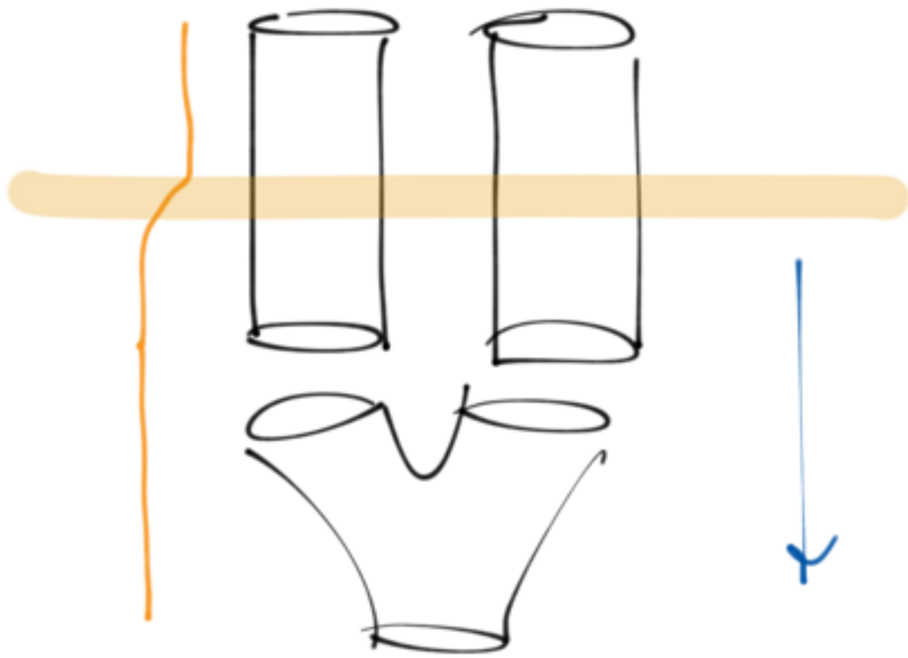
$\overline{M}$  M  

T^*M contains Leg sphere Λ with $A(\Lambda) \cong H_*(\Omega \overline{M})$

The product in symplectic cohomology.

Arguing in a similar way we transport the product on $SH(X)$ to $A^{tto}(\Lambda)$.

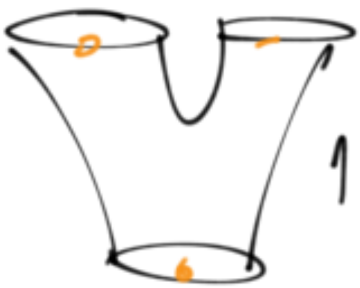
First express the product on $C(X) = \hat{P}(X) \oplus \check{P}(X) \oplus M_0(X)$



The boundary of the parameterized moduli-space gives:

$$\mu(\Phi, \Phi) = \Phi(P) + d\rho + \rho(1 \otimes d + d \otimes 1)$$

so $\mu = P$ on homology.



(In reality also

etc)

We now transport

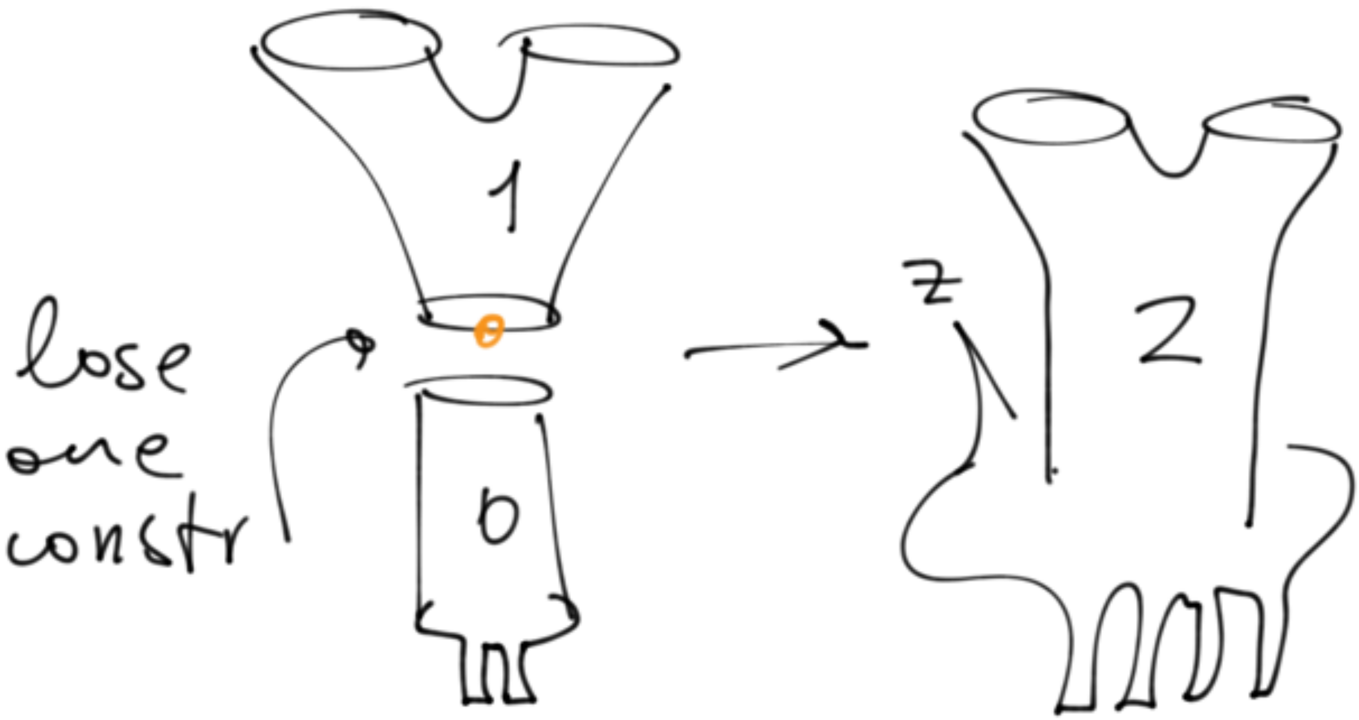
$$P : C(X) \otimes C(X) \longrightarrow C(X)$$

to $A^{H_0}(\Lambda)$ via the surgery isomorphism Φ

<u>Recall</u>

$$\begin{tikzcd}
 & SH(X) \arrow[dl, "\Phi"'] \arrow[d, "\Phi"] \\
A^{H_0}(\Lambda) \arrow[r, "i"] & SH(X_0) \oplus A^{H_0}(\Lambda) \arrow[d, phantom, sloped, "\cong"] \\
 & 0
\end{tikzcd}$$

Consider $\Phi \circ P$:

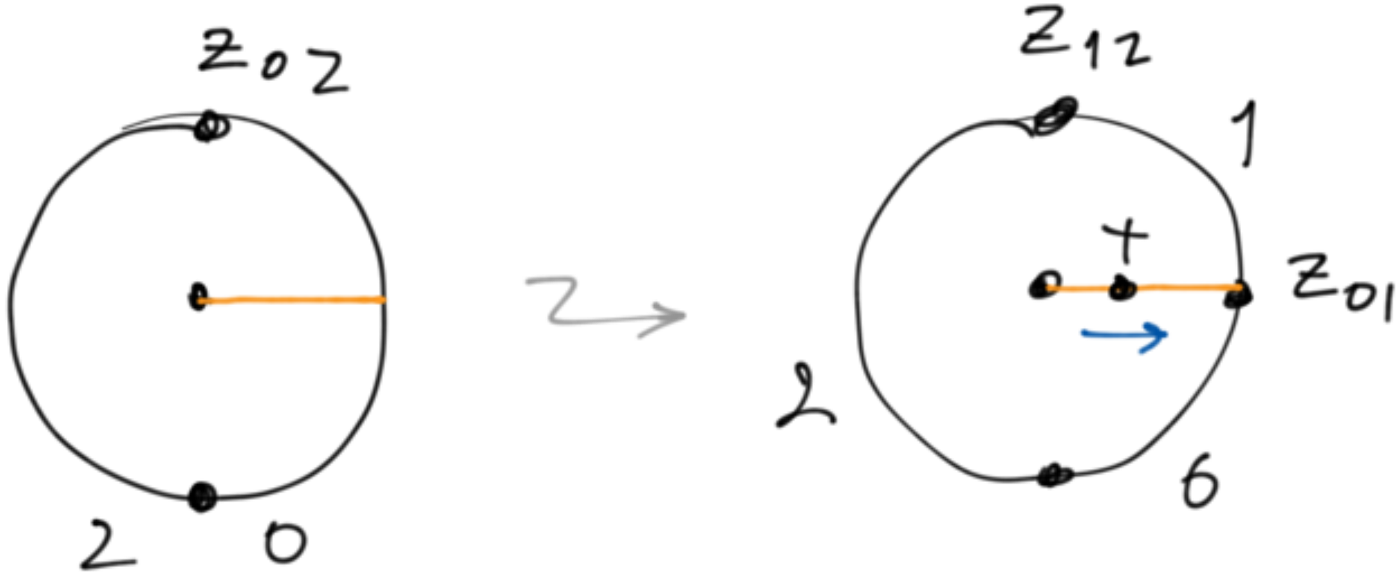


We use 3 copies

L_0, L_1, L_2 and

suppress pure punctures

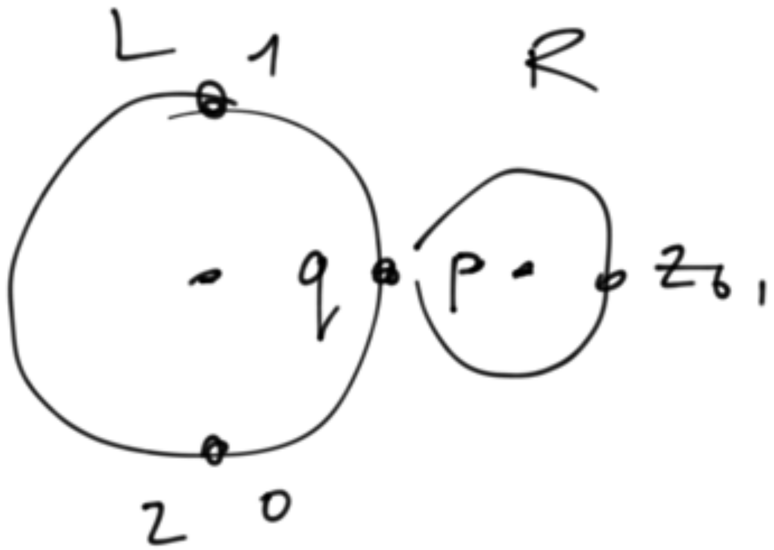
In the domain:



The 2-dim curve is rigidified by fixing the location of 02 and + on the marker line

As we move + there could be splittings which give inessential terms
$d\Psi + \Psi(1\otimes d + d\otimes 1)$.

We get to the split curve

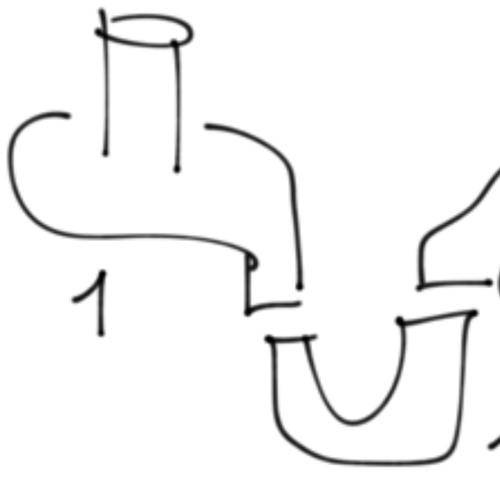


the curve breaks so one component is 2-level

$\dim_{SFT}(L) \geq 1 \quad \dim_{SFT}(R) \geq 0$

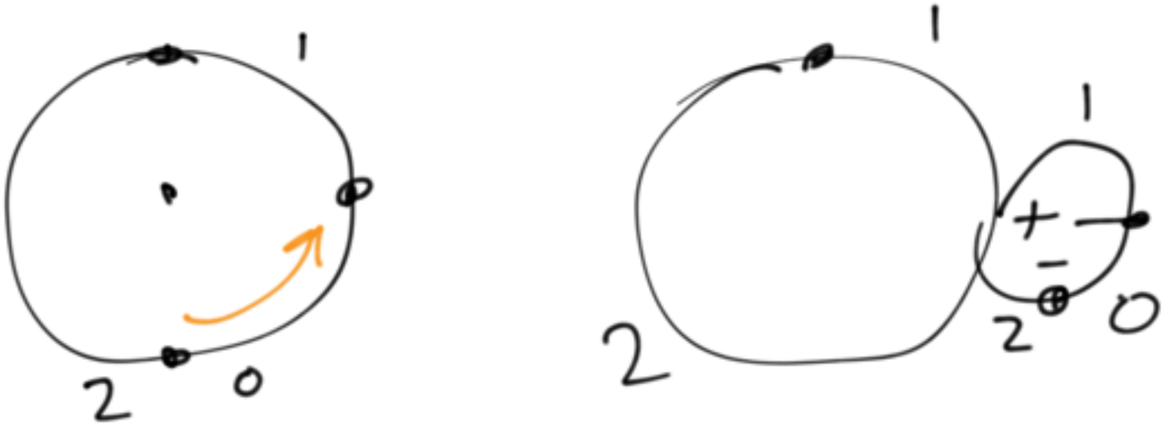
$\dim_{SFT}(\text{tot}) = 2$

$\Rightarrow$

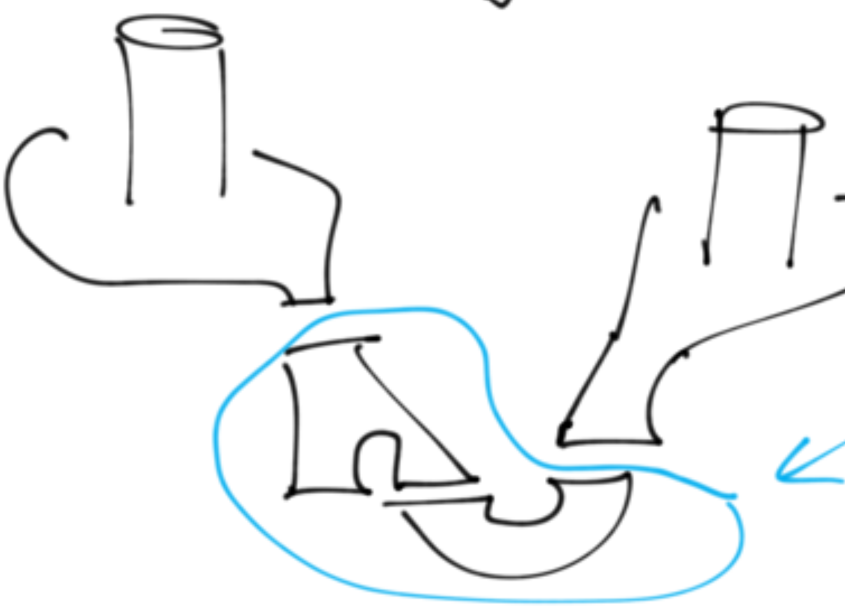


dim = 0

L



$\Downarrow$



product.

One more option:
breaking at z