Topology I
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PROBLEM SET 7
Solutions for #3(a)–(d)

I had a certain solution in mind when I stated Problem 3(c), but on closer inspection, my original idea does not seem to work. (As usual in such situations, everyone who handed in the problem set will get full credit for that problem.) In the solution given below, part (c) is deduced from a purely algebraic proof of (a slight generalization of) part (d). To explain it, I also need to review parts (a) and (b)—that part will just be a repeat of what I said in the Übung.

3. (a) Identify Σ_g with $P/\sim$, where P is the region bounded by a convex polygon large enough to contain the unit disk $\mathbb{D}^2$ in its interior, with $4g$ edges labeled by letters $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$; here the superscript -1 indicates reversed arrows on the corresponding edges. Then $\Sigma_{g,1}$ can be identified with $(P \backslash \mathring{\mathbb{D}}^2)/\sim$. Since $P \backslash \mathring{\mathbb{D}}^2$ admits a deformation retraction to ∂P, there is a deformation retraction of $\Sigma_{g,1}$ to $\partial P/\sim$, which is a wedge sum of $2g$ circles corresponding to the letters $a_1, b_1, \ldots, a_g, b_g$, so

$$\pi_1(\Sigma_{g,1}) \cong \pi_1(\partial P/\sim) \cong F_{\{a_1,b_1,\ldots,a_g,b_g\}}.$$

If $\gamma : S^1 \to \Sigma_{g,1}$ parametrizes $\partial\Sigma_{g,1} = \partial\mathbb{D}^2 \subset (P \backslash \mathring{\mathbb{D}}^2)/\sim$, then the deformation retraction sends γ to the composition of the quotient projection $\partial P \to \partial P/\sim$ with a loop parametrizing ∂P. For a suitable choice of base points, the element of $F_{\{a_1,b_1,\ldots,a_g,b_g\}}$ represented by this loop is

$$[\gamma] = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \in F_{\{a_1,b_1,\ldots,a_g,b_g\}} = \pi_1(\Sigma_{g,1}).$$

This is a nonempty reduced word in the free group $F_{\{a_1,b_1,\ldots,a_g,b_g\}}$, so it is a nontrivial element.

(b) We can write $\Sigma_g = \Sigma_{h,1} \cup_\gamma \Sigma_{k,1}$, which is one way of describing the connected sum $\Sigma_h \# \Sigma_k$ if $\Sigma_{h,1} = \Sigma_h \backslash \mathring{\mathbb{D}}^2$ and $\Sigma_{k,1} = \Sigma_k \backslash \mathring{\mathbb{D}}^2$; note that $g = h + k$. Another way to describe this connected sum in terms of polygons with edges identified was described in Lecture 13: in this picture, $\Sigma_g = P/\sim$ where P has $4g$ edges labeled as in part (a), and γ is a curve through the interior of P leading from the vertex between a_1 and b_g^{-1} to the vertex between a_{h+1} and b_h^{-1}. This curve is homotopic with fixed end points to a curve traversing the boundary of the polygon between those two vertices: in terms of our usual presentation of $\pi_1(\Sigma_g)$ with generators $a_1, b_1, \ldots, a_g, b_g$ and relation $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = e$, it therefore represents

$$[\gamma] = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} \in \pi_1(\Sigma_g).$$

This is a product of the commutators $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$ for $i = 1, \ldots, h$, so it belongs to the commutator subgroup $[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$.

(c) Let us first correct the statement of the question: if $h = 0$ or $k = 0$, then γ is the boundary of a disk in Σ_g and thus clearly represents the trivial element of $\pi_1(\Sigma_g)$. We will show that $[\gamma] \in \pi_1(\Sigma_g)$ is nontrivial if h and k are both positive. In light of part (b), this will follow immediately from the result of part (d) below.

(d) Here is a slightly more general statement: for any proper subset $J \subset \{1, \ldots, g\}$,

$$\prod_{i \in J} [a_i, b_i] \in G := \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = e \right\rangle$$

is a nontrivial element. (It follows incidentally that $\pi_1(\Sigma_g)$ is nonabelian for every $g \geqslant 2$.) In order to prove this, we will define a surjective homomorphism from G to the free group $F_{\{x,y\}}$ on two elements. Choose an element $j \in J$ and another element $\ell \in \{1, \ldots, g\} \backslash J$. Then there is a unique homomorphism

$$\Phi : F_{\{a_1,b_1,\ldots,a_g,b_g\}} \to F_{\{x,y\}}$$

such that

$$a_j \mapsto x,\ b_j \mapsto y,\ a_\ell \mapsto y, b_\ell \mapsto x, \qquad \text{and} \qquad \Phi(a_i) = \Phi(b_i) = e \text{ for all } i \notin \{j, \ell\}.$$

This homormophism is surjective since both of the generators x and y are in its image, and it satisfies

$$\Phi([a_1, b_1] \ldots [a_g, b_g]) = xyx^{-1}y^{-1}yxy^{-1}x^{-1} = e,$$

thus it vanishes on the normal subgroup generated by $\prod_{i=1}^{g}[a_i, b_i]$ and therefore descends to a surjective homomorphism $\Phi : G \to F_{\{x,y\}}$. It also satisfies

$$\Phi\left(\prod_{i \in J}[a_i, b_i]\right) = xyx^{-1}y^{-1} \neq e \in F_{\{x,y\}},$$

implying $\prod_{i \in J}[a_i, b_i] \neq e \in G$.