Intro: What is SFT & what is it good for?

Hamiltonian dynamics: $H \in C^\infty(\mathbb{R}^{2n})$

$(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$

"position" "momentum"

$\rightsquigarrow$ Hamilton's eqns: $\dot{q}_j = \dfrac{\partial H}{\partial p_j}(q,p)$, $\quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}(q,p)$

i.e. $x(t) = (q(t), p(t)) \in \mathbb{R}^{2n}$ satisfies $\dot{x} = X_H(x)$ for

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right).$$

ex: $H(q,p) = \underset{\substack{\uparrow \\ \text{kinetic}}}{\tfrac{1}{2}|p|^2} + \underset{\substack{\uparrow \\ \text{potential}}}{V(q)}$ = total energy $\Rightarrow$ $\dot{q} = p$, $\ddot{q} = \dot{p} = -\nabla V(q)$

Newton's 2nd law

<u>geometrize</u>: Let $\omega_{std} := \sum_{j=1}^{n} dp_j \wedge dq_j \in \Omega^2(\mathbb{R}^{2n})$ ("standard symplectic form on $\mathbb{R}^{2n}$")

ω_{std} is (i) <u>closed</u>: $d\omega_{std} = 0$

(ii) <u>nondegenerate</u>: $\forall\, x \in \mathbb{R}^{2n}$, $\nexists\, X \neq 0 \in T_x\mathbb{R}^{2n}$ s.t. $\omega_{std}(X,Y) = 0 \;\; \forall\, Y \in T_x\mathbb{R}^{2n}$

(ii) $\Rightarrow$ ω_{std} determines an iso $T_x\mathbb{R}^{2n} \longrightarrow T_x^*\mathbb{R}^{2n}$: $X \longmapsto \omega_{std}(X,\cdot)$

<u>EX</u>: The "Hamiltonian vec. fld" X_H is uniquely characterized by

$$\boxed{\omega_{std}(X_H, \cdot) = -dH}$$

<u>defn</u>: a <u>symplectic</u> <u>form</u> (symp. structure) on a smooth $2n$-mfd M is a (i) closed & (ii) nondegenerate 2-form $\omega \in \Omega^2(M)$. (M,ω) is then a <u>symplectic</u> <u>mfd</u>.

Then any smooth fn $H : M \longrightarrow \mathbb{R}$ $\rightsquigarrow$ Ham. vec. fld X_H s.t.

$$\omega(X_H, \cdot) = -dH.$$

prop: The flow $\varphi_t : M \to M$ of X_H preserves (1) H & (2) ω.

pf: (1) $dH(X_H) = -\omega(X_H, X_H) = 0$.

(2) Lie derivative $\mathcal{L}_{X_H}\omega \underset{\text{(Cartan)}}{=} d\underbrace{\iota_{X_H}\omega}_{\substack{\shortparallel \\ \omega(X_H,\cdot) = -dH}} + \iota_{X_H}\underbrace{d\omega}_{=0} = -d(dH) = 0$.

$\Rightarrow \varphi_t^*\omega = \omega \ \forall t$ (i.e. φ_t is a symplectomorphism $(M,\omega) \to (M,\omega)$).

rk: $\omega \in \Omega^2(M^{2n})$ nondeg. $\Leftrightarrow \omega^n := \underbrace{\omega \wedge \dots \wedge \omega}_{n}$ is a volume form on M.

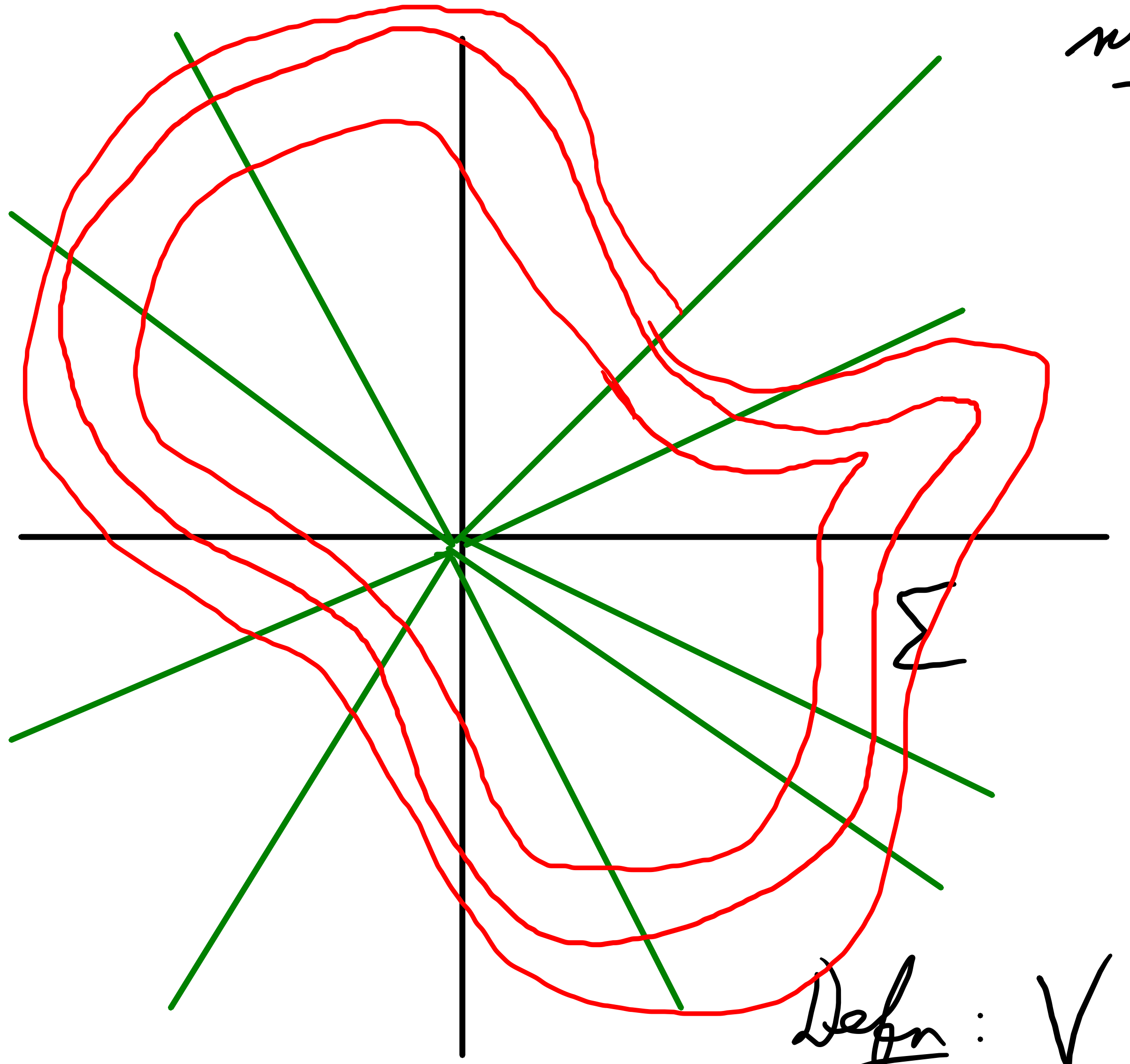
cor: Hamiltonian flows preserves volume on M w.r.t. ω^n.

Q: For $H: M \to \mathbb{R}$ & $c \in \mathbb{R}$ a regular value of H, does X_H admit a periodic orbit on $H^{-1}(c) =: \Sigma$?

rk: answer deps. on Σ but not H. ω nondeg. $\Rightarrow \forall x \in \Sigma$, $\exists!$ 1-dim. subspace $l_x \subseteq T_x\Sigma$ s.t. $\forall X \in l_x$, $\omega(X, \cdot)|_{T_x\Sigma} = 0$, i.e. $l = \ker(\omega|_{T\Sigma})$.

$\omega(X_H, \cdot)|_{T\Sigma} = -dH|_{T\Sigma} = 0 \Rightarrow X_H$ is a section of the subbdl $l \subseteq T\Sigma$ (the characteristic line field). $\therefore$ Up to parametrization, orbits depend on l, not H.

thm (Rabinowitz / Weinstein '78): In $(\mathbb{R}^{2n}, \omega_{std})$, every star-shaped hypersurface has a closed orbit.



motivation: Defn vec. fld $V(q,p) = \frac{1}{2}\sum_{j=1}^{n}\left(q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j}\right)$

so $V \pitchfork \Sigma$. Then

$$\omega_{std}(V, \cdot) = \frac{1}{2}\sum_{j=1}^{n}(p_j\, dq_j - q_j\, dp_j) =: \lambda_{std},$$

$d\lambda_{std} = \omega_{std}$. Then $\mathcal{L}_V \omega_{std} = d\underbrace{\iota_V\omega_{std}}_{\lambda_{std}} + \iota_V \underbrace{d\omega_{std}}_{=0}$

$= \omega_{std}$.

Defn: $V \in \mathfrak{X}(M)$ is a Liouville vector fld on (M,ω) if $\mathcal{L}_V\omega = \omega$; equivalently:
- $(\varphi_V^t)^*\omega = e^t\omega$ for the flow φ_V^t of V
- $\lambda := \omega(V, \cdot)$ is a primitive of ω &

$$\mathcal{L}_V\lambda = d\underbrace{\iota_V\lambda}_{=0} + \iota_V d\lambda = \iota_V\omega = \lambda, \implies (\varphi_V^t)^*\lambda = e^t\lambda.$$

defn: A hypersurface Σ in (M,ω) is of contact type if a nbhd of Σ admits a Liouville vec. fld V s.t. $V \pitchfork \Sigma$.

Then $(-\varepsilon,\varepsilon)\times\Sigma \xrightarrow{\Psi} M : (r,x) \longmapsto \varphi_V^r(x)$ parametrizes a nbhd of Σ & $(\varphi_V^t)^*\lambda = e^t\lambda \Rightarrow$ for $\alpha := \lambda|_{T\Sigma}$, we have $\Psi^*\lambda = e^r\alpha$,

$\Rightarrow \Psi^*\omega = d(e^r\alpha) = e^r(dr\wedge\alpha + d\alpha)$

$\Rightarrow \Psi^*\omega$ restricts to each hypersurfaces $\{r\}\times\Sigma$ as $e^r d\alpha$

$\Rightarrow$ they all have the <u>same</u> char. line field.

$\Rightarrow$ if one of the hyps $\Psi(\{r\}\times\Sigma)\subseteq M$ has a closed orbit, they all do.

$\rightsquigarrow$ <u>Weinstein conjecture</u>: Closed ctct-type hypersurfaces always admit periodic orbits.

<u>status</u>: - For $\dim\Sigma = 3$, proved in 2007 by Taubes (via Seiberg-Witten)

- Otherwise open except some special cases (e.g. using SFT).

rk: Liouville vec. flds are very non-unique: e.g. $V \pitchfork \Sigma$ & $H: M \to \mathbb{R}$, $V + \varepsilon X_H$ is another one for $\varepsilon \in \mathbb{R}$ small.

But $\{V \in \mathfrak{X}(\text{nbhd of } \Sigma) \mid \mathcal{L}_V \omega = \omega \text{ \& } V \pitchfork \Sigma\}$ is convex.

EX: For V Liouville near $\Sigma^{2n-1} \subseteq M^{2n}$, $\lambda := \iota_V \omega$ & $\alpha := \lambda|_{T\Sigma}$,

$V \pitchfork \Sigma \iff \alpha \in \Omega^1(\Sigma)$ satisfies $\alpha \wedge (d\alpha)^{n-1} \neq 0$ everywhere on Σ.

defn: For M^{2n-1} oriented, $\alpha \in \Omega^1(M)$ is a (positive) contact form if $\alpha \wedge (d\alpha)^{n-1} > 0$. The co-oriented hyperplane distribution $\xi := \ker \alpha$ is the associated contact structure on M.

(M, ξ) is called a contact mfd.

A diffeo $\varphi: M_1 \to M_2$ is a contactomorphism $(M_1, \xi_1) \to (M_2, \xi_2)$ if $\varphi_* \xi_1 = \xi_2$ & the co-orientations. If $\xi_j = \ker \alpha_j$ for $j = 1, 2$, this means

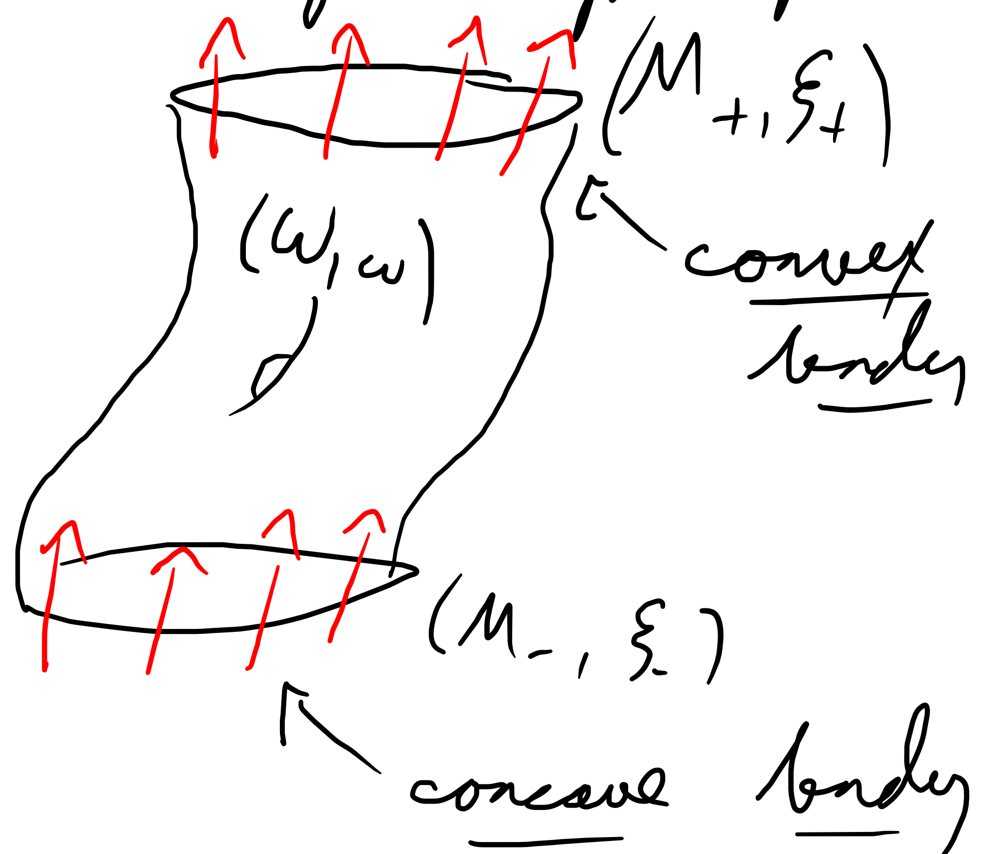
$\varphi^* \alpha_2 = f \alpha_1$ for some smooth fn. $f: M_1 \to (0, \infty)$.

Gray's stability thm: If M^{2n-1} is closed & $\{\xi_s \subseteq TM\}_{s\in[0,1]}$ a smooth 1-param. fam. of ctct strs., then $\xi_s = (\varphi_s)_* \xi_0$ for some smooth 1-param. fam. of diffeos $\varphi_s : M \longrightarrow M$ w/ $\varphi_0 = \mathrm{Id}$. ($\Rightarrow$ all contactomorphic)

(rk: Not true for ctct forms!)

cor: For a closed ctct hyp. $\Sigma \subseteq (M,\omega)$, the ctct str. $\xi := \ker(\iota_V \omega|_{T\Sigma})$ is (up to isotopy) indep. of choice of Liouville vec. fld.

defn: For closed ctct mfds $(M^{2n-1}_\pm, \xi_\pm)$, a symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) is a cpct symp. mfd (W^{2n}, ω) w/ oriented bndry $\partial W = M_+ \amalg \underbrace{(-M_-)}_{\text{orientation reversed}}$ s.t. $M_\pm$ are ctct-type hypersurfaces w/ induced ctct strs. $\xi_\pm$ (up to isotopy).



rk: Bordism theory $\Rightarrow$ $\nexists$ purely topological obstruction to symp. cobordisms betw any 2 closed ctct mfds of same dim.

defn: Cobordism from M_- to M_+

- case $M_- = \emptyset$: symp. filling of M_+ 

- case $M_+ = \emptyset$: symp. cap of M_- 

thm ("soft"): all closed ctct mfds admit symp. caps.

thm ("hard" – via SFT): $\exists$ a seq. of nonfillable ctct mfds $\{M_k\}_{k=0}^{\infty}$ s.t. $\exists$ an exact symp. cobs. 

iff $k \leq l$.