WEDNESDAYS:

LECTURE 11:00 - 12:30

ÜBUNG 13:30 - 15:00

Introduction, part 2: history

1985: J-holomorphic curves in symp. mfds (Gromov)

Σ, M cpx mfds $\Rightarrow$ tangent spaces $T_z\Sigma$, $T_x M$ are $\mathbb{C}$-vec. spaces,

a smooth map $u : \Sigma \to M$ is <u>holomorphic</u> iff $\forall\, z \in \Sigma$,

$du(z) : T_z\Sigma \to T_{u(z)} M$ is $\mathbb{C}$-linear, i.e. $\boxed{Tu \circ i = i \circ Tu}$.

<u>defn</u>: An <u>almost cpx structure</u> on M^{2n} is a smooth bndl map

$J : TM \to TM$ s.t. $J^2 = -\mathrm{Id}$. $\rightsquigarrow$ TM becomes a cpx vec. bndl

$(a+ib)X := aX + bJX$. Call (M,J) an <u>almost</u> cpx mfd

(of cpx dim. n).

<u>ex</u>: A <u>Riemann surface</u> is an almost cpx mfd (Σ, j) of cpx dim. 1

($\dim_{\mathbb{R}} \Sigma = 2$).

<u>thm of Gauss</u>: All Riem. surfs. are also cpx. mfds., i.e. $\exists$ an atlas of charts w/ hol. transition maps s.t. j = mult. by i in any hol. coords.

(i.e. all almost cpx strs. in real dim. 2 are integrable — not true in higher dims.)

defn: A ~~smooth~~ map $u : \Sigma \to M$ ((Σ, j) a Riem. surf., (M, J) an almost cpx mfd.)

is a pseudoholomorphic curve (also J-holomorphic) if $\boxed{Tu \circ j = J \circ Tu.}$ $(*)$

nonlinear Cauchy-Riemann eqn.

~~In~~ small nbhds $\mathcal{U} \subseteq \Sigma$,

Gauss $\Rightarrow$ $\exists$ "hol. local coords." $(s,t) : \mathcal{U} \to \mathbb{R}^2$ (equiv. $s+it : \mathcal{U} \to \mathbb{C}$)

s.t. $j(\partial_s) = \partial_t$, $j(\partial_t) = -\partial_s$.

Then $(*) \Longleftrightarrow$ $Tu(\partial_s) + J \circ Tu \circ j(\partial_s) = 0 \Longleftrightarrow \boxed{\partial_s u + J(u)\, \partial_t u = 0}$

1st-order elliptic PDE

nonlinear

elliptic $\Rightarrow$ spaces of sols. up to parametrizations ("moduli space")

~~are~~ want to be
- smooth fin.-dim. mfds of dim = some Fredholm index determined by topology
- compact

sample thm (Gromov '85 + McDuff '89): Spse (M,ω) = closed connected sympl. 4-mfd s.t.

(i) $\nexists$ a sympl. submfd $S^2 \cong S \subseteq M$ w/ $[S]\cdot[S] = -1$ ← homological self-int. #

(ii) $\exists$ 2 sympl. submfds $S^2 \cong S_1, S_2 \subseteq M$ s.t. $[S_1]\cdot[S_1] = [S_2]\cdot[S_2] = 0$

& $S_1 \cap S_2$ is a single transverse positive intersection pt.

Then $(M,\omega) \cong (S^2 \times S^2, \underbrace{\sigma_1 \oplus \sigma_2}_{\text{area forms on } S^2})$.

pf sketch: Defn $\mathcal{J}_\tau(M,\omega) := \{$almost $\mathbb{C}$-strs $J : TM \to TM \mid \omega(X,JX) > 0 \ \forall X \neq 0\}$

"J is tamed by ω"

$\mathcal{J}(M,\omega) := \{ J \in \mathcal{J}_\tau(M,\omega) \mid \omega(X,Y) = \omega(JX,JY) \ \forall X,Y \}$

$\Longleftrightarrow g(X,Y) := \omega(X,JY)$ is a Riemannian metric.

fundamental lemma: $\mathcal{J}_\tau(M,\omega)$ & $\mathcal{J}(M,\omega)$ are always nonempty & contractible.

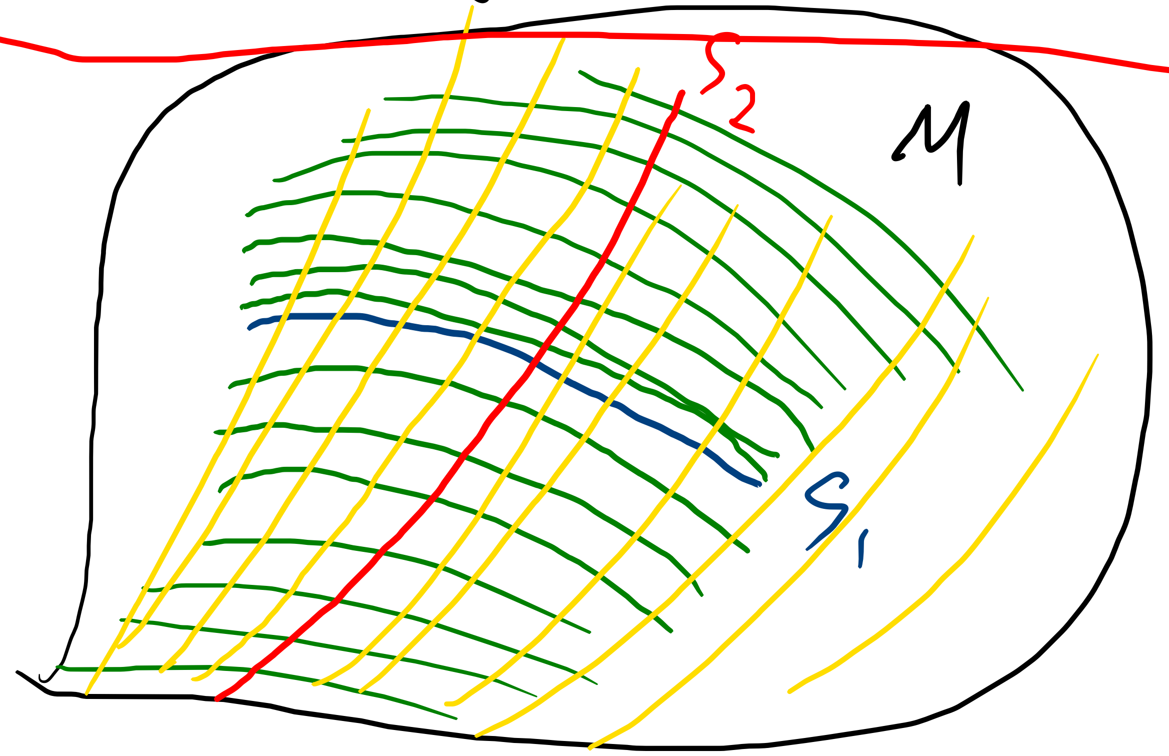
$\Sigma^2 \subseteq (M,\omega)$ a symp. submfd ($\omega|_{T\Sigma}$ also nondeg.)

$\Rightarrow$ can choose $J \in \mathcal{J}(M,\omega)$ s.t. $J(T\Sigma) = T\Sigma$, then $j := J|_{T\Sigma}$ makes (Σ, j) a Riem. surface s.t. inclusion $\Sigma \hookrightarrow M$ is J-hol.

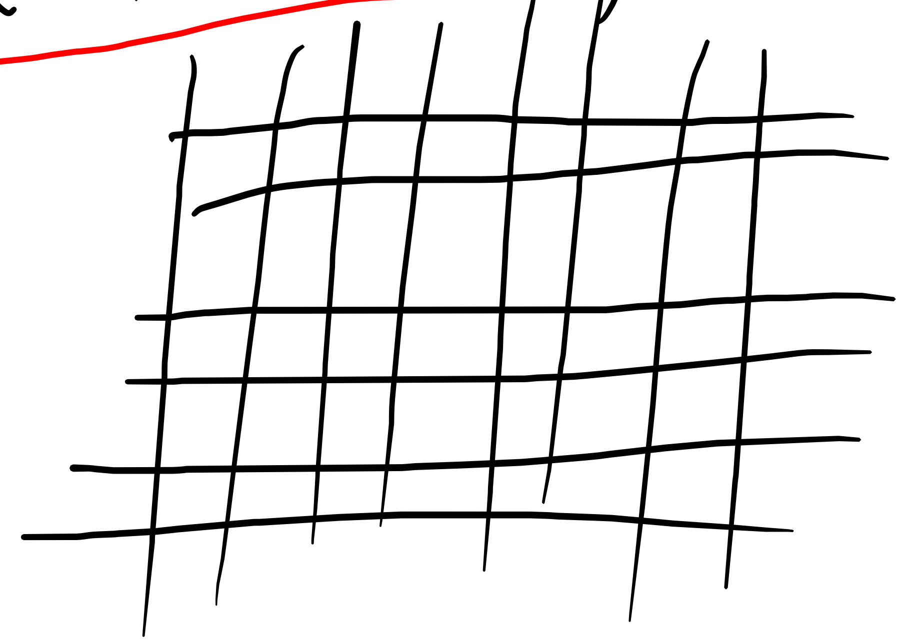
Now choose $J \in \mathcal{J}(M,\omega)$ s.t. S_1 & S_2 are both J-hol. curves in this sense.

For $i = 1,2$, let $\mathcal{M}_i(J)$ be the connected moduli of J-hol. curves containing S_i.

<u>the hard part</u>: $\mathcal{M}_i(J)$ are each cpt oriented 2-dim. mfds consisting of embedded J-hol. spheres that foliate M, & each $u \in \mathcal{M}_1(J)$ intersects each $v \in \mathcal{M}_2(J)$ exactly once, transversely & positively.



$\cong$



$S^2 \times S^2$.

$\square$

~ 1988 : Floer homology

(M^{2n}, ω) closed sympl. mfd., $\{H_t : M \to \mathbb{R}\}_{t \in S^1}$ $\quad (S^1 = \mathbb{R}/\mathbb{Z})$

$\rightsquigarrow$ t-dep. Hamiltonian vec. fld X_{H_t} on M.

Arnold conj : $\#\{1\text{-periodic orbits of } X_{H_t}\} \geq \#\underbrace{\{\text{critical pts. of a fn. } f : M \to \mathbb{R}\}}_{\text{determined by top. of } M}$

Suppose (M, ω) is symplectically aspherical : $\int_{S^2} u^* \omega = 0 \quad \forall\, u : S^2 \to M$.

Then $\exists$ well-def'd action functional

$$\mathcal{A}_H : \underset{\text{contractible}}{C^\infty(S^1, M)} \to \mathbb{R} : \gamma \longmapsto -\int_{\mathbb{D}^2} \bar{\gamma}^* \omega + \int_{S^1} H_t(\gamma(t))\, dt$$

w/ $\bar{\gamma} : \mathbb{D}^2 \to M$ any choice s.t. $\bar{\gamma}|_{\partial \mathbb{D}^2} = \gamma$.

EX : $\mathrm{Crit}(\mathcal{A}_H) = \{\gamma : S^1 \to M \text{ contr. s.t. } \dot{\gamma} = X_{H_t}(\gamma)\}$.

idea : Find crit pts. of $\mathcal{A}_H$ by following "negative gradient flow lines"

$u : \mathbb{R} \to C^\infty(S^1, M)$ s.t. $\partial_s u(s) + \nabla \mathcal{A}_H(u(s)) = 0 \quad (**)$

$u : \mathbb{R} \times S^1 \to M, \quad u(s,t) := u(s)(t)$.

Choose $\{J_t \in \mathcal{J}(M,\omega)\}_{t\in S^1}$, so $\exists$ L^2-product on $\Gamma(\gamma^*TM)$ def'd by

$$\langle \eta, \xi\rangle_{L^2} := \int_{S^1} \omega(\eta(t), J_t(\gamma(t))\,\xi(t))\,dt.$$

<u>EX</u> <u>following</u>: For a family $\{\gamma_\rho \in C^\infty_{\mathrm{contr}}(S^1, M)\}_{\rho\in(-\varepsilon,\varepsilon)}$ w/ $\gamma_0=\gamma$, $\partial_\rho\gamma_\rho|_{\rho=0} = \eta \in \Gamma(\gamma^*TM)$,

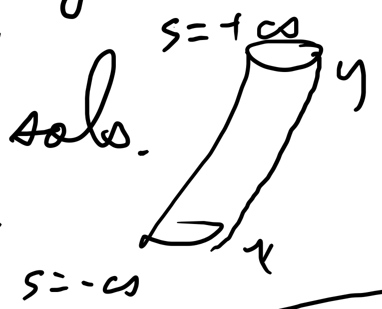
$$\frac{d}{d\rho}\mathcal{A}_H(\gamma_\rho)\Big|_{\rho=0} = \langle J_t(\partial_t\gamma - X_{H_t}(\gamma)), \eta\rangle_{L^2}$$

$\Rightarrow$ can sensibly defn. $\nabla\mathcal{A}_H(\gamma) := J_t(\partial_t\gamma - X_{H_t}(\gamma)) \in \Gamma(\gamma^*TM)$.

(**) becomes $\boxed{\partial_s u + J_t(u)\,\partial_t u - J_t(u)\,X_{H_t}(u) = 0}$ "Floer eqn." (inhomogeneous nonlinear CR-eqn.)

<u>deep</u> <u>insight</u> (Floer): $\exists$ a chain cpx CF_*, freely generated by contractible 1-per. orbits $x: S^1 \to M$ s.t.

$$\partial\langle x\rangle = \sum_{\text{suitable } y} \#\underbrace{\left(\text{sols. to } (**)\,/\,\mathbb{R}\right)}_{\mathcal{M}(x,y)}\langle y\rangle$$



Its homology $\cong H_*(M)$.

<u>reason for $\partial^2 = 0$</u>: In cases where $\dim\mathcal{M}(x,y) = 1$, $\mathcal{M}(x,y)$ has a natural compactification consisting of "broken" Floer cylinders:

$\Rightarrow$ The coeff. of $\langle y\rangle$ in $\partial^2\langle x\rangle$ counts broken Floer cyls. from y to x = bdry of a cpt 1-mfd $\overline{\mathcal{M}(x,y)}$.

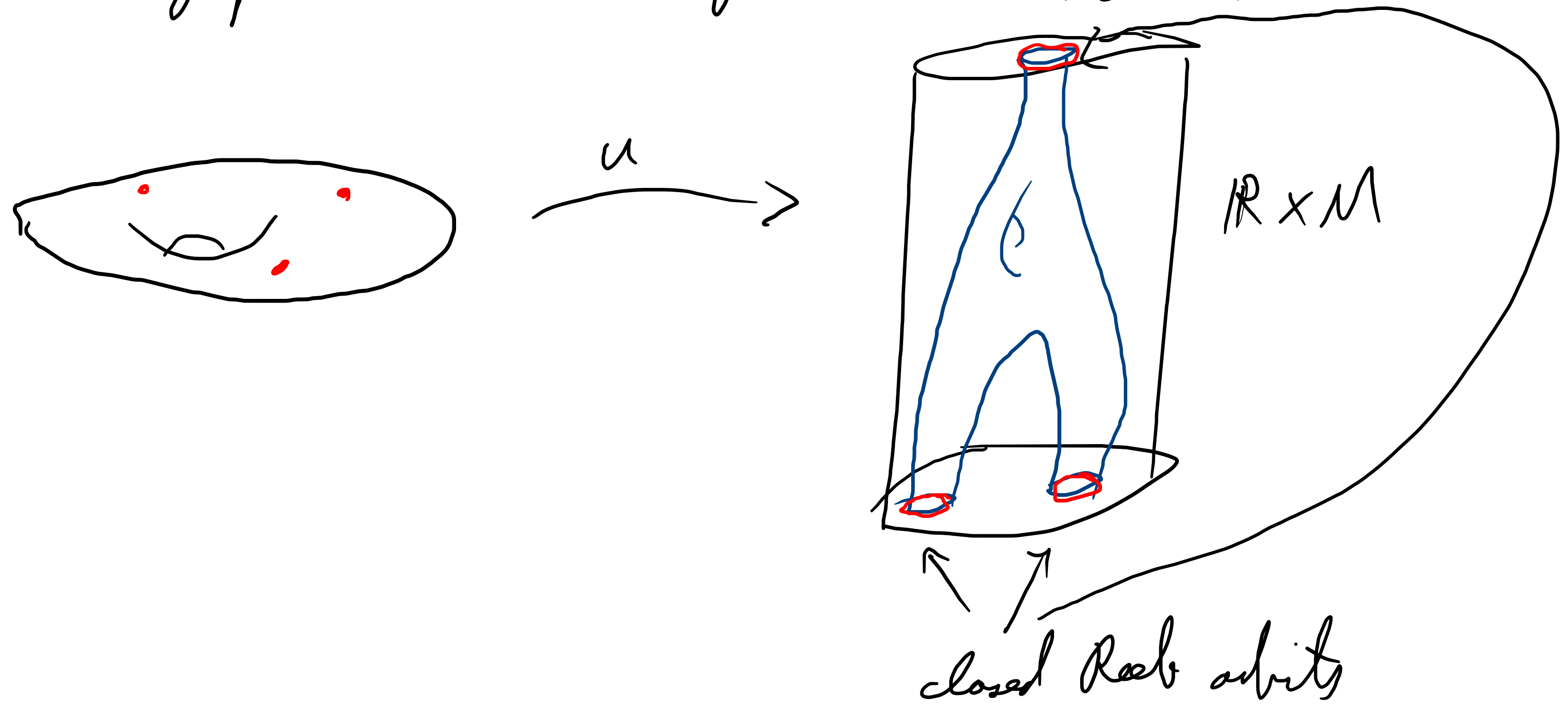
<u>1993</u>: <u>Hofer's approach to the Weinstein conj.</u>

$(M^{2n-1}, \xi = \ker\alpha)$ a ctct mfd: is also a ctct-type hypersurface in its own <u>symplectization</u>: $(\underset{r}{\mathbb{R}} \times M, d(e^r\alpha))$. Char. line fld on each $\{r\} \times M$ is spanned by the <u>Reeb vector field</u> $R_\alpha \in \mathfrak{X}(M)$: $\begin{cases} d\alpha(R_\alpha, \cdot) \equiv 0 \\ \alpha(R_\alpha) \equiv 1 \end{cases}$.

<u>thm</u>: For a natural class of compatible a.c.s.'s J on $(\mathbb{R} \times M, d(e^r\alpha))$, any <u>punctured</u> J-hol. curve $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ w/ <u>finite energy</u>

"closed Riem. surf. \ {finite subset}"

is asymptotic at each puncture to a closed Reeb orbit.



<u>2000</u>: Eliashberg-Givental-Hofer introducing SFT:

defn. a Floer-type homological invt of ctct mfds (w/ maps induced by symp. cobordism) by counting finite-energy punctured J-hol. curves in symplectization (or in the "completion" of a symp. cob.).

<u>rk</u>: $\underset{(s,t)}{\mathbb{R} \times S^1}$ w/ its natural a.c.s. $j(\partial_s) = \partial_t$ $\cong \mathbb{C} \setminus \{0\} = S^2 \setminus \{0, \infty\}$.