Basics of J-hol. curves

(Σ, j) = Riem. surf., (W, J) = almost cpx mfd of dim. $2n$

approximate goal: $\mathcal{M}(J) := \{ u \in C^\infty(\Sigma, W) \mid Tu \circ j = J \circ Tu + \text{more conditions} \}$

is a fin.-dim. mfd. (for generic choices) w/ a natural compactification.

" strategy":

(1) defn. an ∞-dim. Banach mfd $\mathcal{B} \subseteq \{\text{maps } \Sigma \to M\}$ containing $\mathcal{M}(J)$.

(2) defn a Banach space bndl $\mathcal{E} \to \mathcal{B}$ & a smooth section

$\bar\partial_J : \mathcal{B} \to \mathcal{E}$ s.t. $\mathcal{M}(J) = \bar\partial_J^{-1}(0)$.

candidate: $\mathcal{B} \ni u \longmapsto \bar\partial_J(u) := du + J(u) \circ du \circ j \in \underbrace{\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW))}_{= \Omega^{0,1}(\Sigma, u^*TW)} \subseteq \mathcal{E}_u$.

(3) $u \in \bar\partial_J^{-1}(0) \rightsquigarrow$ linearization $D_u := D\bar\partial_J(u) : T_u\mathcal{B} \to \mathcal{E}_u$

prove D_u is a Fredholm operator, compute its index

(4) Prove for generic J, $\bar\partial_J \pitchfork (0\text{-section of } \mathcal{E} \to \mathcal{B})$ ($\Leftrightarrow \forall\, u \in \bar\partial_J^{-1}(0)$, D_u is surjective)

Then I.F.T. $\Rightarrow$ $\bar\partial_J^{-1}(0) \subseteq \mathcal{B}$ is a smooth fin.-dim. submfd w/

$T_u\mathcal{M} = \ker D_u \subseteq T_u\mathcal{B}$, hence $\dim \mathcal{M}(J)$ near u is $\operatorname{ind} D_u$.

What is D_u?

Given $u \in \bar{\partial}_J^{-1}(0)$, consider 1-param. fam. $\{u_\epsilon \in \bar{\partial}_J^{-1}(0)\}_{\epsilon \in (-\varepsilon,\varepsilon)}$ w/ $u_0 = u$,

$\partial_\epsilon u_\epsilon|_{\epsilon=0} =: \eta \in \Gamma(u^*TW)$. In local hol. coords (s,t) on Σ,

$\partial_s u_\epsilon + J(u_\epsilon)\partial_t u_\epsilon = 0 \Rightarrow$ for any connection ∇ on W,

$0 = \nabla_\epsilon [\partial_s u_\epsilon + J(u_\epsilon)\partial_t u_\epsilon]\big|_{\epsilon=0}$; since $\partial_s u + J(u)\partial_t u = 0$, RHS is indep. of choice of ∇.

Assume ∇ is symmetric, so $\nabla_\epsilon \partial_s = \nabla_s \partial_\epsilon$ etc...

$$\Rightarrow 0 = \nabla_s \eta + J(u)\nabla_t \eta + (\nabla_\eta J)\partial_t u = \underbrace{(\nabla\eta + J(u)\circ\nabla\eta\circ j + (\nabla_\eta J)\,du\circ j)}_{=0}(\partial_s)$$

$\rightsquigarrow$ <u>defn</u>. The linearized Cauchy-Riemann operator $D_u : \Gamma(u^*TW) \to \Omega^{0,1}(\Sigma, u^*TW)$ for $u \in \bar{\partial}_J^{-1}(0)$ is given by $\boxed{D_u\eta = \nabla\eta + J(u)\nabla\eta\circ j + (\nabla_\eta J)\,du\circ j}$ for any symm. conn. ∇.

<u>defn</u>: a <u>linear</u> <u>Cauchy-Riemann</u> <u>type</u> op. on a cpx vec. bndl E over (Σ, j) is any 1st-order $\mathbb{R}$-linear diff. op. $D : \Gamma(E) \to \Omega^{0,1}(\Sigma, E) := \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E))$ satisfying $\boxed{D(f\eta) = (\bar{\partial} f)\eta + f D\eta}$ $\forall\, \eta \in \Gamma(E)$, $f \in C^\infty(\Sigma, \mathbb{R})$, where $\bar{\partial} f := df + i\, df\circ j \in \Omega^{0,1}(\Sigma)$.

<u>ex</u> 1: D_u is a lin. CR-op.

<u>EX 2</u>: The <u>difference</u> btwn any 2 lin. CR-ops is a $\mathbb{R}$-lin. bndl map $A : E \to \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$, i.e. "0th-order term."

<u>cor</u>: In suitable local coords. & trivs., every linear CR-op. is locally equivalent to $\bar{\partial} + A : C^\infty(\mathbb{D}, \mathbb{C}^m) \to C^\infty(\mathbb{D}, \mathbb{C}^m)$, for

$\bar{\partial} := \partial_s + i\,\partial_t$ in coords $s + it \in \mathbb{D}$ (unit disk $\subset \mathbb{C}$), $A : \mathbb{D} \xrightarrow{C^\infty} \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$.

Sobolev spaces: assume $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$ w/ "reasonable" bndry.

Fns f, g on U, we say $\partial_j f = g$ *weakly* ("in the *sense* of *distributions*")

if $\forall$ "test fns" $\varphi \in C_0^\infty(U)$, $\int_U g\varphi = -\int_U f\,\partial_j \varphi$.

$W^{k,p}(U) := \{ f \in L^p(U) \mid f$ has weak derivs. in $L^p(U)$ of all orders up to $k\}$

$$\|f\|_{W^{k,p}} := \sum_{\substack{\alpha \text{ multiindices} \\ |\alpha| \le k}} \|\partial^\alpha f\|_{L^p}$$

$\left(\text{for } \alpha = (\alpha_1, \dots, \alpha_n),\ \partial^\alpha f := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f,\ |\alpha| := \alpha_1 + \dots + \alpha_n.\right)$

thm ($\Leftarrow$ standard L^p-theory): $W^{k,p}(U)$ is a separable Banach space $\forall$ $k \ge 0$, $1 \le p < \infty$, & it contains $W^{k,p}(U) \cap C^\infty(U)$ as a dense subspace.

intuition: $f \in W^{k,p}$ has "$k - \frac{n}{p}$ contin. derivatives".

useful Sobolev estimates: $\exists$ natural contin. maps

Sobolev emb. thm + Rellich cptness thm

(1) If $kp > n$: $W^{k,p}(\mathcal{U}) \hookrightarrow C^0(\overline{\mathcal{U}}) \Rightarrow W^{k+d,p}(\mathcal{U}) \hookrightarrow C^d(\overline{\mathcal{U}})$ for $d \geq 0$.

cpct if $\mathcal{U}$ bdd.

(2) If $k \geq m$, $p \leq q$ & $k - \frac{n}{p} \geq m - \frac{n}{q}$, $W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U})$,

cpct if $\mathcal{U}$ bdd & ineq. is strict.

appendix A

(3) If $kp > n$ & $k - \frac{n}{p} \geq m - \frac{n}{q}$, $W^{k,p} \times W^{m,q} \to W^{m,q} : (f,g) \mapsto fg$.

In particular, $W^{k,p}$ is a Banach algebra: $\|fg\|_{W^{k,p}} \leq c \|f\|_{W^{k,p}} \cdot \|g\|_{W^{k,p}}$.

(4) If $kp > n$, $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^m$, $C^k(\Omega, \mathbb{R}^N) \times W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : (f,u) \mapsto f \circ u$

$W^{k,p}(\mathcal{U}, \Omega) := \{ u \in W^{k,p}(\mathcal{U}, \mathbb{R}^m) \mid \overline{u(\mathcal{U})} \subseteq \Omega \}$

elliptic regularity theory

ÜBUNG

Set $\bar\partial := \partial_s + i\partial_t$, $\partial := \partial_s - i\partial_t$ on $\mathbb{D} \ni s+it$,

$$K(z) := \frac{1}{2\pi z}, \quad K \in L^1_{loc}(\mathbb{C}).$$

fundamental elliptic estimate:

(1) $\forall\, p \in (1,\infty)$, $\bar\partial : W^{1,p}(\mathbb{D}) \longrightarrow L^p(\mathbb{D})$ has a bdd right inverse

$T : L^p \longrightarrow W^{1,p}$ given by $Tf = K * f$ for $f \in C_0^\infty(\mathring{\mathbb{D}})$,

i.e. $$(Tf)(z) = \int_{\mathbb{C}} \frac{f(\zeta)}{2\pi(z-\zeta)}\, dm(\zeta)$$

↑ Lebesgue measure w.r.t. $\zeta \in \mathbb{C}$.

(2) $\exists$ const $c > 0$ s.t. $\forall\, f \in W_0^{k,p}(\mathring{\mathbb{D}}) := W^{k,p}$-closure of $C_0^\infty(\mathring{\mathbb{D}})$,

$$\|f\|_{W^{k,p}} \le c\,\|\bar\partial f\|_{W^{k-1,p}}.$$

<u>application</u>: linear local existence

<u>thm</u>: Assume $A \in L^p(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$ for some $p > 2$. Then

$$\begin{cases} (\bar\partial + A)u = 0 \\ u(0) = u_0 \end{cases}$$ has weak sols. $u \in W^{1,p}(\mathbb{D}_\varepsilon, \mathbb{C}^m)$ $\forall\, u_0 \in \mathbb{C}^m$

$\mathbb{D}_\varepsilon = \{z \in \mathbb{C} \mid |z| \le \varepsilon\}$

if $\varepsilon > 0$ small enough.

<u>pf</u>: Consider $\Phi_\varepsilon : W^{1,p}(\mathbb{D}) \to L^p(\mathbb{D}) \times \mathbb{C}^m : u \mapsto ((\bar\partial + \chi_{\mathbb{D}_\varepsilon} A)u,\ u(0))$

w/ $\chi_{\mathbb{D}_\varepsilon}$:= char. fn. of $\mathbb{D}_\varepsilon$. <u>Note</u>: $p > 2 \Rightarrow W^{1,p} \hookrightarrow C^0 \Rightarrow u(0)$ well-def'd & $|u(0)| \le \|u\|_{C^0} \le c\|u\|_{W^{1,p}}$.

Then $\|(\Phi_\varepsilon - \Phi_0)u\|_{L^p \times \mathbb{C}^m} = \|\chi_{\mathbb{D}_\varepsilon} A u\|_{L^p(\mathbb{D})} = \|Au\|_{L^p(\mathbb{D}_\varepsilon)} \le \|A\|_{L^p(\mathbb{D}_\varepsilon)} \cdot \|u\|_{C^0}$

$\le c \underbrace{\|A\|_{L^p(\mathbb{D}_\varepsilon)}}_{\to 0 \text{ as } \varepsilon \to 0} \cdot \|u\|_{W^{1,p}}$ $\qquad \|\Phi_\varepsilon - \Phi_0\| \to 0$ as $\varepsilon \to 0$.

$\bar\partial$ has a bdd right-inverse $\Rightarrow \Phi_0$ also does $\Rightarrow$ so does Φ_ε for $\varepsilon > 0$ suff. small. Choose $u \in \Phi_\varepsilon^{-1}(0, u_0)|_{\mathbb{D}_\varepsilon}$. ∎

__cor__ (similarity principle): For $D : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ a lin. CR-op.,
every $z_0 \in \Sigma$ admits a nbhd with a __continuous__ triv. of E that identifies any given local sol. $D\eta = 0$ with a hol. fn.

$\Rightarrow$ __cor__ (unique continuation): a nontrivial sol. to $D\eta = 0$ cannot vanish to ∞ order at any pt.

__pf of similarity__: WLOG $z_0 = 0 \in \mathbb{D}$, $\eta : \mathbb{D} \to \mathbb{C}^n$ satisfies $(\bar\partial + A)\eta = 0$ for some $A : \mathbb{D} \xrightarrow{C^\infty} \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Choose $C : \mathbb{D} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ s.t.
$C\eta = A\eta$, & $C \in L^p$ for $p > 2$. Then $\exists$ local sol. $e_1, \dots, e_m$ to $(\bar\partial + C) e_j = 0$ near 0 s.t. they form the std $\mathbb{C}$-basis at 0.
$W^{1,p} \hookrightarrow C^0 \Rightarrow$ they also form a basis on $\mathbb{D}_\varepsilon$ for $\varepsilon > 0$ small.

Now $\eta = \sum_{j=1}^m f_j e_j$ for some fns $f_j : \mathbb{D}_\varepsilon \to \mathbb{C}$.

$(\bar\partial + C)\eta = (\bar\partial + C) e_j = 0$ & $\bar\partial + C$ satisfies Leibniz rule for fns in $C^\infty(\Sigma, \mathbb{C})$,

$\Rightarrow \bar\partial f_j = 0$. ∎