rk (from yesterday): All $f \in H^1(S^1)$ are absolutely contin.

Lemma 1: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, X, Y Banach spaces over $\mathbb{F}$, $k \in \mathbb{N}$, $i \in \mathbb{Z}$.

$\mathrm{Fred}^{i,k}_{\mathbb{F}} := \{ T \in \mathcal{L}_{\mathbb{F}}(X,Y) \mid \dim_{\mathbb{F}} \ker T = k \ \& \ \mathrm{codim}_{\mathbb{F}} \operatorname{im} T = k - i \} \subseteq \mathcal{L}_{\mathbb{F}}(X,Y)$

is an analytic ($\Rightarrow$ smooth) submfd of $\mathrm{codim}_{\mathbb{F}} = k(k-i)$, &

$X^{i,k} := \{ (T, x) \mid T \in \mathrm{Fred}^{i,k}_{\mathbb{F}}, \ x \in \ker T \}$ is a smooth vector bndl.

Pf: Given $T_0 \in \mathrm{Fred}^{i,k}_{\mathbb{F}}$, $\exists$ splittings $X = \left\{ \begin{matrix} V & \xrightarrow[\cong]{T_0} & W \\ \oplus & & \oplus \\ K & & C \end{matrix} \right\} = Y$, $K = \ker T_0$, $W = \operatorname{im} T_0$, $\dim C = k - i$

$\Rightarrow T_0 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ for an iso. $A_0 : V \xrightarrow{\cong} W$.

Choose a nbhd $\mathcal{O} \subseteq \mathcal{L}_{\mathbb{F}}(X,Y)$ of T_0 s.t. $\forall\ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{O}$, A is invtbl.

$\Phi : \mathcal{O} \longrightarrow \mathrm{Hom}_{\mathbb{F}}(K, C) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto D - CA^{-1}B$.

$\Psi : \mathcal{O} \longrightarrow \mathcal{L}_{\mathbb{F}}(X) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}$ (invtbl)

Then $T\Psi(T) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & \Phi(T) \end{pmatrix}$ has kernel $\{0\} \oplus \ker \Phi(T)$

$\Psi(T) : X \xrightarrow{\cong} X$ sends $\ker \Phi(T) \subseteq \ker T_0 \subseteq X$ isomorphically to $\ker T$. $\quad V \oplus K = X$

$\Rightarrow \dim_{\mathbb{F}} \ker T = k$ iff $\Phi(T) = 0$.

Similarly, $\mathrm{codim}_{\mathbb{F}} \operatorname{im} T = k - i$ iff $\Phi(T) = 0$.

$d\Phi(T_0) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \delta \Rightarrow d\Phi(T_0)$ surj $\Rightarrow \Phi^{-1}(0)$ is a submfd near T_0,

with $\mathrm{codim} = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(\ker T_0, \operatorname{coker} T_0) = k(k-i)$

Then $\forall\ T \in \Phi^{-1}(0)$, $\Psi(T) : \ker T_0 \xrightarrow{\cong} \ker T$ defns a local triv. of $X^{i,k}$. ☒

variations:

(2) $X = H^1(S^1)$, $Y = L^2(S^1)$. $\mathrm{Fred}_{\mathbb{R}}^{sym,k} := \{ T \in \mathcal{L}_{\mathbb{R}}^{sym}(H^1; L^2) \mid \dim\ker T = \operatorname{codim}\operatorname{im} T = k \}$.

Symmetry $\Rightarrow \langle x, T_0 y \rangle_{L^2} = \langle T_0 x, y \rangle_{L^2} = 0 \quad \forall\ y \in H^1$ if $x \in \ker T_0$,

since $\operatorname{codim}\operatorname{im} T_0 = \dim\ker T_0$, this $\Rightarrow L^2 = \operatorname{im} T_0 \oplus \ker T_0$.

$\Rightarrow$ Can defn. $C := K = \ker T_0$, $V := \operatorname{im} T_0 \cap H^1$.

$\rightsquigarrow \Phi : \mathcal{O} \longrightarrow \mathrm{End}_{\mathbb{R}}(K)$, T symmetric iff $\Phi(T) \in \mathrm{End}_{\mathbb{R}}^{sym}(K)$.

$\Rightarrow \mathrm{Fred}_{\mathbb{R}}^{sym,k} \subseteq \mathcal{L}_{\mathbb{R}}^{sym}(H^1; L^2)$ is a submfd of codim $= \dim \mathrm{End}_{\mathbb{R}}^{sym}(K) = \dfrac{k(k+1)}{2}$.

Case $k=1$: $\exists$ canonical iso. $\mathrm{End}_{\mathbb{R}}^{sym}(K) = \mathbb{R}$.

$$cI \longleftarrow c$$

$\Rightarrow \mathrm{Fred}_{\mathbb{R}}^{sym,1}$ has a canonical co-orientation.

(3) For $T_0 \in \mathcal{L}_{\mathbb{R}}^{\mathrm{sym}}(H^1, L^2)$ Fredholm of index 0, $\lambda_0 \in \sigma(T_0) \Rightarrow$

$T_0 - \lambda_0 \in \mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym}, k}$ for $k := \dim \underbrace{\ker(T_0 - \lambda_0)}_{=: K}$ = "multiplicity of λ_0"

$T \in \mathcal{L}_{\mathbb{R}}^{\mathrm{sym}}(H^1, L^2)$ near T_0, $\lambda \in \mathbb{R}$ near $\lambda_0 \Rightarrow$

$\lambda \in \sigma(T)$ iff $\Phi(T - \lambda) = 0 \in \mathrm{End}_{\mathbb{R}}^{\mathrm{sym}}(K)$. $\lambda \longmapsto \det \Phi(T - \lambda)$ is analytic (of same mult.)

$\Rightarrow$ (i) $\sigma(T)$ is discrete

(ii) $\exists$ nbhds $\mathcal{O} \subseteq \mathcal{L}_{\mathbb{R}}^{\mathrm{sym}}(H^1, L^2)$ of T_0 & $\mathcal{U} \subseteq \mathbb{R}$ of λ_0 s.t.

$\forall\, T \in \mathcal{O}$, T has exactly k e-vals in $\mathcal{U}$ (counted w/ multiplicity).

(iii) If λ_0 is simple (i.e. $\dim K = 1$), $\exists$ smooth fn. $\mathcal{O} \longrightarrow \mathcal{U} : T \longmapsto \lambda(T) \in \sigma(T)$.

(iv) For $\{T_s\}_{s \in (-\epsilon, \epsilon)}$ a smooth path & $\lambda(s) \in \sigma(T_s)$ simple e-vals w/ $\lambda(0) = 0$,

$\lambda'(0) \neq 0 \Leftrightarrow$ intersection of $s \longmapsto T_s$ with $\mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym}, 1}$ is transverse,

& $\operatorname{sign} \lambda'(0)$ = sign of the intersection (w.r.t. canonical co-orientation).

(4) $\{A_s := -i\partial_t - S_s(t)\}_{s\in[-1,1]}$ path of asymp. ops.

(i) For $\lambda(s)\in\sigma(A_s)$ simple e-vals, $|\lambda'(s)| \le \|\partial_s S_s\|_{\mathcal{L}(L^2)}$.

Pf: Lemma 1 $\Rightarrow$ $\exists$ smooth fam. of e-fns $\eta(s)\in\ker(A_s-\lambda(s))\subseteq H^1$, WLOG $\|\eta(s)\|_{L^2}=1$. Then $\lambda'(s) = \partial_s\langle \eta(s), A_s\eta(s)\rangle_{L^2}$

$$= \underbrace{-\langle \eta(s), (\partial_s S_s)\eta(s)\rangle_{L^2}}_{|\ \ |\le \|\partial_s S_s\|_{\mathcal{L}(L^2)}\cdot\|\eta(s)\|^2_{L^2}} + 2\lambda(s)\underbrace{\langle\partial_s\eta(s),\eta(s)\rangle_{L^2}}_{=\frac{1}{2}\partial_s\langle\eta(s),\eta(s)\rangle_{L^2}=0}$$

∎

(ii) codim $\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{R}}\ge 3$ $\forall$ $k\ge 2$ $\Rightarrow$ $(-1,1)\times\mathbb{R}\to \mathcal{L}^{\mathrm{sym}}_{\mathbb{R}}(H^1,L^2): (s,\lambda)\mapsto A_s-\lambda$ "should" not hit it.

Sard-Smale thm. (see Appendix C) $\Rightarrow$ after a C^∞-small pert. of $\{S_s\}_{s\in(-1,1)}$ can assume $\sigma(A_s)$ for $-1<s<1$ has only simple e-vals., & $s\mapsto A_s$ intersects $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{R}}$ transversely.

Pf of spectral flow thm:

Given $\{A_s\}_{s\in[-1,1]}$, perturb as in 4(ii)

3(iii) + 4(i) $\Rightarrow \exists$ smooth bdd fns. $\{\lambda_j : (-1,1) \longrightarrow \mathbb{R}\}$ s.t.

$\ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ & $\sigma(A_s) = \{\lambda_j(s) \mid j \in \mathbb{Z}\}$.

Only fin.-many can cross 0, always $\pitchfork$. (3)(ii) $\Longrightarrow$ they extend contin. to $[-1,1]$ & count multiplicity at $s = \pm 1$. $\square$

defn: $A = -i\partial_t - S$ w/ $\ker A = \{0\}$ (i.e. A is nondegenerate),
has Conley-Zehnder index

$$\mu_{CZ}(A) := -\mu^{spec}(A_0, A) \quad \text{where} \quad A_0 := -i\partial_t - \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{\mathbb{R}^{2n\times 2n}}$$

$\left(i = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\right)$

($\iff$ $\mu_{CZ}(A_0) = 0$ &

$$\mu_{CZ}(A_-) - \mu_{CZ}(A_+) = \mu^{spec}(A_-, A_+).)$$

For $(E, J, \omega) \to S^1$ a Hermitian vec. bdl w/ trivialization τ, defn.

$$\mu^\tau_{CZ}(A) := \mu_{CZ}(A_\tau) \quad \text{for } A_\tau := A \text{ in any unitary htpic } \tau.$$

For $\gamma : S^1 \to M$ a Reeb orbit & τ a triv. of $\gamma^*\xi$, $\mu^\tau_{CZ}(\gamma) := \mu^\tau_{CZ}(A_\gamma)$.

thm: 2 nondeg. asymp. ops $A_\pm$ are htpic through a family of **nondeg** ops. A_s $\iff$ $\mu^{spec}(A_-, A_+) = 0$, i.e. $\mu^\tau(A_-) = \mu^\tau(A_+)$.

pf: Picture

EX: On a direct sum bndl, $\mu_{CZ}^{\tau_1 \oplus \tau_2}(A_1 \oplus A_2) = \mu_{CZ}^{\tau_1}(A_1) + \mu_{CZ}^{\tau_2}(A_2)$.

thm: For $n = 1$, $A = -i\partial_t - S(t)$, $\lambda \in \sigma(A)$ have eigenspaces $E_\lambda \subseteq H^1(S^1, \mathbb{C})$.
$\exists$ a well-def'd monotone incr. fn. $\sigma(A) \to \mathbb{Z} : \lambda \mapsto \mathrm{wind}(e_\lambda)$ for any nontrivial $e_\lambda \in E_\lambda$,
& it takes all values exactly twice (counting multiplicity).

pf: Consider $A_0 := -i\partial_t$, so $A_0\eta = \lambda\eta \iff \dot{\eta} = i\lambda\eta \iff \eta(t) = \eta(0)e^{i\lambda t}$,
$\lambda = 2\pi k$ for $k = \mathrm{wind}(\eta) \in \mathbb{Z}$ $\Rightarrow$ thm is true for A_0.

General A: deform A_0 to A, draw a picture.

<u>thm</u>: For $n=1$ & A nondeg.,

$$\mu_{CZ}(A) = 2\alpha_-(A) + p(A) = 2\alpha_+(A) - p(A)$$

where $\alpha_+(A) := \min_{\lambda>0} \operatorname{wind}(e_\lambda)$, $\alpha_-(A) := \max_{\lambda<0} \operatorname{wind}(e_\lambda)$

$p(A) := \alpha_+(A) - \alpha_-(A) \in \{0,1\}$ "<u>parity</u>".

<u>pf</u>: For $A_0 := -i\partial_t - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, check explicitly.

For general A, choose generic htpy $A_0 \rightsquigarrow A$, check all terms in formula change the same way when e-vals. cross 0. $\square$