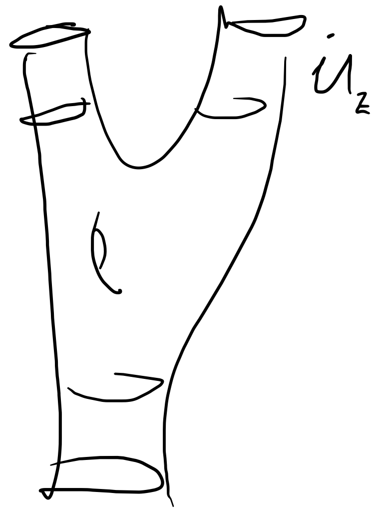


$$\begin{array}{c} E \\ \downarrow \\ \dot\Sigma = \Sigma \setminus \Gamma \end{array}$$

$D : W^{k,p}(E) \longrightarrow W^{k-1,p}(F) \quad (F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot\Sigma, E))$

CR-op. of class C^m, asymp at ends to asymp. ops $\{A_z\}_{z\in\Gamma}$, all nondeg.

<u>to prove</u>: D is Fredholm w/ index α, kernel indep. of k, p.

<u>Lemmas</u>: (1) $\|\eta\|_{W^{k,p}(\Sigma_0)} \leq c\|D\eta\|_{W^{k-1,p}(\Sigma_1)} + c\|\eta\|_{W^{k-1,p}(\Sigma_1)}$ for $\Sigma_0 \overset{\text{open}}{\subseteq} \bar\Sigma_0 \subseteq \Sigma_1 \overset{\text{open}}{\subseteq} \bar\Sigma_1 \overset{\text{cpt}}{\subseteq} \dot\Sigma$.

(2) $\eta \in L^p(E)$ & $D\eta \in W^{k-1,p}(F) \Rightarrow \eta \in W^{k,p}(E)$.

(3) $D = \partial_s - A$ on $\mathbb{R}\times S^1$ for A nondeg., D is an iso.

<u>Lemma 4</u>: For D on $Z_+^R := [R,\infty)\times S^1 \subseteq \mathcal{U}_z$ for $z \in \Gamma^+$ (sim. if $z\in\Gamma^-$), if A_z is nondeg., then $\forall\, R \gg 1$, $\exists\, c > 0$ s.t.

$$\|\eta\|_{W^{k,p}(Z_+^R)} \leq c\|D\eta\|_{W^{k-1,p}(Z_+^R)} \quad \forall\, \eta \in W_0^{k,p}(\mathring{Z}_R^+).$$

<u>rk</u>: $\eta \in W^{k,p}(\mathring{Z}_R^+)$ is in $W_0^{k,p}(\mathring{Z}_R^+) \Leftrightarrow$ its extension to $\mathbb{R}\times S^1$ with $\eta = 0$ outside Z_+^R is also in $W^{k,p}$.

<u>Pf</u>: Write $D = \bar\partial + S(s,t)$, $D_0 := \bar\partial + S_\infty(t)$ s.t. $\|S - S_\infty\|_{C^{k-1}(Z_+^R)} \xrightarrow{R\to\infty} 0$.

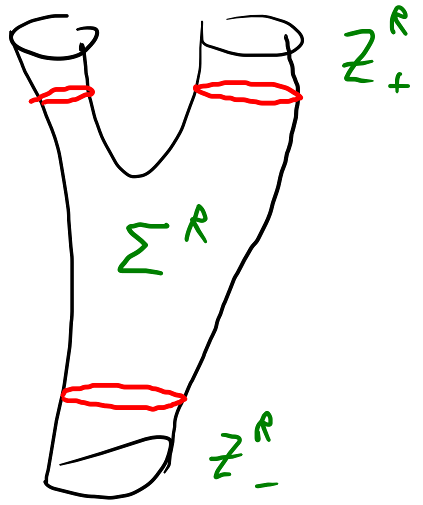
Every $\eta \in W_0^{k,p}(\mathring{Z}_+^R)$ has canonical extension to $\mathbb{R}\times S^1$,

$$\Rightarrow \|\eta\|_{W^{k,p}(Z_+^R)} = \|\eta\|_{W^{k,p}(\mathbb{R}\times S^1)} \overset{(3)}{\leq} c\|D_0\eta\|_{W^{k-1,p}(\mathbb{R}\times S^1)} = c\|D_0\eta\|_{W^{k-1,p}(Z_+^R)}$$

$$\leq c\|D\eta\|_{W^{k-1,p}(Z_+^R)} + c\underbrace{\|(S-S_\infty)\eta\|_{W^{k-1,p}(Z_+^R)}}_{\leq c'\underbrace{\|S-S_\infty\|_{C^{k-1}(Z_+^R)}}_{\leq \frac{1}{2c'} \text{ for } R\gg 1}\cdot\|\eta\|_{W^{k,p}(Z_+^R)}}$$

$$\Rightarrow \frac{1}{2}\|\eta\|_{W^{k,p}(Z_+^R)} \leq c\|D\eta\|_{W^{k-1,p}(Z_+^R)}.$$

$\square$



Lemma 5: Assume all A_z nondeg., let $\Sigma^R := \dot\Sigma \setminus \bigcup_{z\in\Gamma^\pm} Z_\pm^R$

Then $\forall R >> 1$, $\exists\, c>0$ s.t.

$$\|\eta\|_{W^{k,p}(\dot\Sigma)} \le c\|D\eta\|_{W^{k-1,p}(\dot\Sigma)} + c\|\eta\|_{W^{k-1,p}(\Sigma^R)}.$$

note: $W^{k,p}(\dot\Sigma) \longrightarrow W^{k-1,p}(\Sigma^R) : \eta \longmapsto \eta|_{\Sigma^R}$ factors through

$W^{k,p}(\Sigma^R) \hookrightarrow W^{k-1,p}(\Sigma^R)$ w/ $\overline{\Sigma^R}$ cpt, $\Rightarrow$ a cpt op.

$\Rightarrow$ **cor**: $\dim \ker D < \infty$ & $\operatorname{im} D$ closed.

Pf: Choose $\beta \in C_0^\infty(\Sigma^R)$ s.t. $\beta|_{\Sigma^{R-1}} \equiv 1$. so $(1-\beta)\eta \in W_0^{k,p}(Z_\pm^{R-1})$ on each end.

$$(*) \quad \|\beta\eta\|_{W^{k,p}(\dot\Sigma)} = \|\beta\eta\|_{W^{k,p}(\Sigma^R)} \overset{(1)}{\le} c\|D(\beta\eta)\|_{W^{k-1,p}(\Sigma^{R+1})} + c\|\beta\eta\|_{W^{k-1,p}(\Sigma^{R+1})}$$

$$\le c'\|D\eta\|_{W^{k-1,p}(\dot\Sigma)} + c'\|\eta\|_{W^{k-1,p}(\Sigma^{R+1})}.$$

$$(**) \quad \|(1-\beta)\eta\|_{W^{k,p}(\dot\Sigma)} = \sum_{z\in\Gamma} \|(1-\beta)\eta\|_{W^{k,p}(Z_\pm^{R-1})} \overset{(4)}{\le} \sum_{z\in\Gamma} c\|D[(1-\beta)\eta]\|_{W^{k-1,p}(Z_\pm^{R-1})}$$

$$\le c'\|D\eta\|_{W^{k-1,p}(\dot\Sigma)} + c'\|\eta\|_{W^{k-1,p}(\Sigma^R)}$$

(absorbed $\|1-\beta\|_{C^{k-1}}$ & $\|\bar\partial\beta\|_{C^{k-1}}$ into const. $c'>0$).

$\bar\partial\beta \neq 0$ only $(R-1,R)\times S^1$.

$$\|\eta\|_{W^{k,p}(\dot\Sigma)} \le \|\beta\eta\| + \|(1-\beta)\eta\| = (*) + (**). \qquad \blacksquare$$

exponential decay: Assume $m \geq 1$.

intuition from Morse homology: For $D = \partial_s - A(s)$ for fns $\mathbb{R} \to \mathbb{R}^n$

$\lim_{s\to\infty} A(s) = A_+$ symm. + invtbl. $\Rightarrow$ eqn. $D\eta = 0$ for $s >> 0$

in a suitable triv. is close to $\dot\eta(s) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \eta(s)$.

Then $\forall$ sols. w/ $\lim_{s\to\infty} \eta(s) = 0$, $\eta(s) \leq C e^{-\lambda s}$ for any $\lambda > 0$ s.t.

$\sigma(A_+) \cap [-\lambda, \lambda] = \emptyset$.

trick: $\|A_+ v\| \geq \lambda \|v\|$ $\forall v$. Can assume also true for $A(s)$ $\forall s \geq R >> 1$.

Def $\gamma(s) := \frac{1}{2}\|\eta(s)\|^2 = \frac{1}{2}\langle \eta(s), \eta(s)\rangle$, then $\dot\eta = A\eta$ $\Rightarrow$

$\dot\gamma = \langle \eta, A\eta\rangle$, $\quad \ddot\gamma = \langle A\eta, A\eta\rangle + \langle \eta, \dot A \eta\rangle + \langle \eta, AA\eta\rangle$

$= 2\|A\eta\|^2 + \langle \underbrace{(A^T - A)}_{\text{small}}\eta, A\eta\rangle + \langle \eta, \underbrace{\dot A}_{\text{small}}\eta\rangle$

$\Rightarrow \forall s \geq R >> 1$, $\quad \ddot\gamma \geq 2\lambda^2\|\eta\|^2 - c\|\eta\|^2$ for $c > 0$ arb. small

$= 4\lambda^2\gamma - c'\gamma \geq 4\lambda^2\gamma$ after small adjustment to λ.

Compare $\gamma(s)$ with $\alpha(s) := \gamma(R) e^{-2\lambda(s-R)}$,

$\begin{cases} \ddot\gamma \geq 4\lambda^2\gamma \\ \ddot\alpha = 4\lambda^2\alpha \end{cases}$ $\quad \alpha(R) = \gamma(R)$ $\quad \Rightarrow f := \gamma - \alpha$ satisfies $\begin{cases} \ddot f \geq 4\lambda^2 f \\ f(R) = 0. \end{cases}$

EX (MVT): Unless $\lim_{s\to\infty} f(s) = \infty$, $f \leq 0$ $\forall s \geq R$.

$\Rightarrow \|\eta(s)\|^2 \leq C e^{-2\lambda s}$ $\forall s \geq R$, i.e. $\|\eta(s)\| \leq C' e^{-\lambda s}$.

$\Rightarrow$ Lemma 6: For $D = \bar\partial + S(s,t)$ on $Z_\pm$ of class C^m $(m \geq 1)$ asymp. to

$A = -i\partial_t - S_\infty(t)$, if $\lambda > 0$ s.t. $\sigma(A) \cap [-\lambda, \lambda] = \emptyset$,

$\forall u \in L^p(Z_\pm)$ $(2 \leq p \leq \infty)$ satisfying $Du = 0$, $\exists c > 0$ s.t.

$\|u(s,\cdot)\|_{L^2(S^1)} \leq c e^{\mp\lambda s}$. ☒

note: If $1 < p < 2$, regularity (lemma 2) $\Rightarrow u \in W^{1,p} \overset{\text{Sobolev}}{\Longrightarrow} u \in L^q$ for

some $q \geq 2$, so lemma valid $\forall$ $1 < p \leq \infty$.

Now exp. decay $\Rightarrow u \in L^p(Z_+)$ $\forall p \in (1,\infty) \overset{\text{reg.}}{\Longrightarrow} u \in W^{m+1,p}(Z_+)$.

cor: If all A_z nondeg., then $\ker D \subseteq \bigcap_{k \leq m+1} \bigcap_{1<p<\infty} W^{k,p}(E)$.

formal adjoint: Fix suitable area forms on $\dot{\Sigma}$ & bndl metrics $\langle\, ,\rangle_E$, $\langle\, ,\rangle_F$

$\rightsquigarrow$ L^2-pairings $\langle \eta, \xi\rangle_{L^2} := \mathrm{Re}\int_{\dot{\Sigma}} \langle \eta, \xi\rangle_E \, d\mathrm{vol}$ for $\eta, \xi \in \Gamma(E)$,
sim. on F.

$D^* :=$ the ! 1st-order diff. op. $C^{m+1}(F) \to C^m(E)$ s.t.

$$\langle \lambda, D\eta\rangle_{L^2} = \langle D^*\lambda, \eta\rangle_{L^2} \quad \forall \quad \eta \in C_0^{m+1}(E), \quad \lambda \in C_0^{m+1}(F).$$

rk: $C_0^{m+1} \subseteq W^{k,p}$ dense $\Rightarrow$ $\langle \lambda, D\eta\rangle_{L^2} = \langle D^*\lambda, \eta\rangle_{L^2}$ also holds for $\eta \in W^{1,p}$
$\lambda \in W^{1,q}$ if $\frac{1}{p} + \frac{1}{q} = 1$.

EX: In suitable local trivs., $D^* = -\partial + S$
for $\partial := \partial_s - i\partial_t$ & S of class C^m.

globally: Let $\bar{E} :=$ same real bndl as E but w/ $\mathbb{C}$-str. $\bar{J} := -J$.
Then $\exists$ $\mathbb{C}$-bndl iso. $\bar{E} \xrightarrow{\cong} E^*: v \longmapsto \langle v, \cdot\rangle_E$. Sim.
$E \cong \Lambda^{0,1}E^* = \overline{\mathrm{Hom}}_{\mathbb{C}}(E, \mathbb{C}) \cong \bar{E}^*$.

$\Rightarrow F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E) = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, \mathbb{C}) \otimes_{\mathbb{C}} E \cong T\dot{\Sigma} \otimes_{\mathbb{C}} E$ (via bndl metric $\langle\, ,\rangle_{T\dot{\Sigma}} := d\mathrm{vol}(\cdot, j\cdot)$)

$\Rightarrow \mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F) \cong (T\dot{\Sigma})^* \otimes_{\mathbb{C}} F \cong \underbrace{(T\dot{\Sigma})^* \otimes_{\mathbb{C}} T\dot{\Sigma}}_{\text{trivial: } \lambda \otimes v \mapsto \lambda(v)} \otimes_{\mathbb{C}} E \cong E$

$\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F) = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, \bar{F})$.

thm: These natural isos. det'd by our bndl metrics & area form identify D^* w/ a CR-type op. on $\bar{F}$ of class C^m, C^m-asymptotic at each end U_z to $-\bar{A}_z$ (i.e. $-A_z$ acting on $\bar{E}_z$)

A_z nondeg $\Leftrightarrow$ $-\bar{A}_z$ nondeg

cor: If all A_z nondeg., D^* is also semi-Fredholm & has

$$\ker D^* \subseteq \bigcap_{k \le m+1} \bigcap_{1 < p < \infty} W^{k,p}(F). \quad \boxtimes$$

Lemma 7: (i) $W^{k-1,p}(F) = \operatorname{im} D \oplus \ker D^*$, (ii) $W^{k-1,p}(E) = \operatorname{im} D^* \oplus \ker D$.

($\Rightarrow$ $\operatorname{coker} D \cong \ker D^*$, $\operatorname{coker} D^* \cong \ker D$ $\Rightarrow$ D is Fredholm & dim ker & dim coker are indep. of k, p.)

pf of (i) for $k=1$ (rest follows by regularity)

claim 1: $W^{k-1,p}(F) = \operatorname{im} D + \ker D^*$.

pf: $\operatorname{im} D$ closed, $\dim \ker D^* < \infty$ $\Rightarrow$ $\operatorname{im} D + \ker D^*$ is closed.

If $\neq L^p$, Hahn-Banach thm $\Rightarrow \exists\ \lambda \in (L^p(F))^* \cong L^q(F)$

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ s.t. $\lambda \neq 0$ but $\langle \lambda, D\eta + \alpha \rangle_{L^2} = 0 \quad \forall\ \eta \in W^{1,p}$, $\alpha \in \ker D^*$.

$\Rightarrow \langle \lambda, D\eta \rangle_{L^2} = 0 \quad \forall\ \eta \in W^{1,p} \Rightarrow \lambda$ is a weak sol. to $D^*\lambda = 0$, $\lambda \in L^q$

$\overset{\text{(reg.)}}{\Rightarrow} \lambda \in \ker D^*$, $\langle \lambda, \alpha \rangle_{L^2} = 0$ for $\alpha = \lambda \in \ker D^*$ $\Rightarrow \lambda = 0$ contra!

claim 2: $\operatorname{im} D \cap \ker D^* = \{0\}$. Else $\exists\ \eta \in W^{1,p}(E)$ s.t.

$D\eta \in \ker D^*$, then $D\eta \in L^p \Rightarrow D\eta \in L^q$ for $\frac{1}{p} + \frac{1}{q} = 1$.

$\Rightarrow \langle D\eta, D\eta \rangle_{L^2} = \langle \eta, D^*(D\eta) \rangle_{L^2} \overset{\eta \in W^{1,q}}{=} 0 \Rightarrow D\eta = 0$. $\square$