index thm: $E \to \dot{\Sigma}$ asymp. Hermitian VB of cpx rank n,

D a CR-type op. on E of class C^m $(1 \le m \le \infty)$, C^m-asymptotic to nondeg. asymp. ops. $\{A_z\}_{z \in \Gamma}$ at the punctures $z \in \Gamma^+ \cup \Gamma^-$.

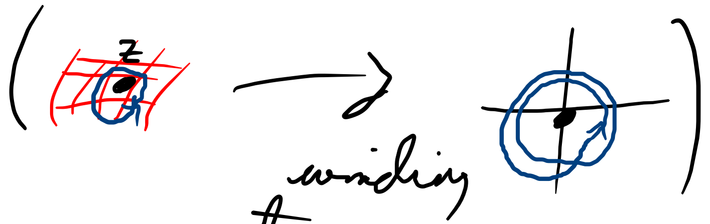
Then
$$\boxed{\operatorname{ind}(D) = n\chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z \in \Gamma^+} \mu_{CZ}^\tau(A_z) - \sum_{z \in \Gamma^-} \mu_{CZ}^\tau(A_z)}$$

for any choice of asymp. trivializations τ.

defn: The relative 1st Chern number $c_1^\tau(E) \in \mathbb{Z}$ of $E \to \dot{\Sigma}$ w.r.t. τ is uniquely det'd by:

(1) If $n = 1$, $c_1^\tau(E) := \sum_{z \in \eta^{-1}(0)} \underbrace{\operatorname{ord}(\eta; z)}_{\text{order of the zero}}$ for any choice of section $\eta \in \Gamma(E)$

order of the zero: (



)

s.t. $\eta^{-1}(0)$ is finite & $\eta = \text{const.}$ near punctures w.r.t. τ.

EX: $c_1^\tau(E)$ deps. on τ up to htpy, but not on η.

(2) $c_1^{\tau_1 \oplus \tau_2}(E_1 \oplus E_2) = c_1^{\tau_1}(E_1) + c_1^{\tau_2}(E_2)$.

EX: Every $\mathbb{C}$-VB over a R.S. is a direct sum of line bundles.

rk: If $\Gamma = \emptyset$, $c_1^\tau(E) = \langle \underbrace{c_1(E)}_{\text{1st Chern class } H^2(\Sigma)}, [\Sigma] \rangle$.

EX: RHS of index formula is indep. of τ.

simplifying assumptions:

(1) $m = \infty$, (2) We consider $D : W^{k,p}(E) \longrightarrow W^{k-1,p}(\overset{\overset{\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)}{\shortparallel}}{F})$ only for $k=1, p=2$,

i.e. $D : H^1(E) \longrightarrow L^2(F)$.

(3) $n = 1$: rest follows via direct sum property

idea of pf (due to Taubes):

Consider large $\mathbb{C}$-antilinear 0th-order perturbation: $B : E \longrightarrow F$ $\mathbb{C}$-antilinear,

$D_r := D + rB$ for $r \gg 0$.

If $\Gamma = \emptyset$, $\mathrm{ind}(D) = \mathrm{ind}(D_r)$ $\forall\, r \in \mathbb{R}$.

random Q: Ar how many pts. must B vanish?

$B \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F))$, $\# B^{-1}(0) = c_1(\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F)) = c_1(\mathrm{Hom}_{\mathbb{C}}(\bar{E},F))$

$= c_1(\bar{E}^* \otimes F) = c_1(E \otimes F) = c_1(E) + c_1(\underset{\overset{\shortparallel}{\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E) \,\hat{=}\, T\Sigma \otimes \bar{E}}}{F}) = c_1(E) + \underbrace{c_1(T\Sigma)}_{\overset{\shortparallel}{\chi(\Sigma)} \text{ (Poincaré-Hopf)}} + c_1(E)$

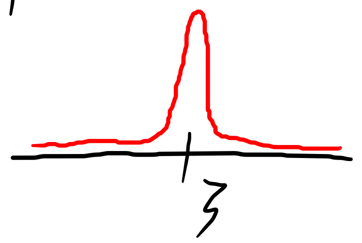
$= \chi(\Sigma) + 2c_1(E) \overset{(!)}{=} \mathrm{ind}\, D$.

Coincidence?

thm 2: Spse $B \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F))$ has a finite zero-set $\underbrace{Z_+(B)}_{\text{order} = +1} \amalg \underbrace{Z_-(B)}_{-1} \subseteq \dot{\Sigma}$

all w/ order ± 1. Then for $D_r := D + rB$ for $r \gg 0$

$\exists \begin{cases} \text{for each } \zeta \in Z_+(B) \text{ a section } \eta_\zeta \in \Gamma(E) \text{ concentrated near } \zeta \\ \quad '' \quad \zeta \in Z_-(B) \quad '' \quad \xi_\zeta \in \Gamma(F) \quad '' \quad '' \end{cases}$



s.t. $\ker D_r = \mathrm{Span}_{\mathbb{R}} \{\eta_\zeta\}_{\zeta \in Z_+(B)}$, $\ker D_r^* = \mathrm{Span}_{\mathbb{R}} \{\xi_\zeta\}_{\zeta \in Z_-(B)}$.

cor: For $r \gg 0$, $\mathrm{ind}\, D_r$ = algebraic count of zeroes of B.

main tool: Weitzenböck formula

defn: $D : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, E)) = \Omega^{1,0}(\dot{\Sigma}, E)$ is an anti-CR-type op.

if $\forall\ f \in C^\infty(\dot{\Sigma}, \mathbb{R})$, $\eta \in \Gamma(E)$, $D(f\eta) = \partial f \cdot \eta + f D\eta$

where $\partial f := df - i \cdot df \circ j$

rk 1: $\partial f = 0 \iff f$ is antiholomorphic, i.e. $df \circ j = -i\, df$.

f, g antihol. $\Rightarrow$ fg antihol. $\Rightarrow$ $\exists$ notion of antihol. VBs.

rk 2: locally, anti-CR op. $D = \partial + A$ for $\partial := \partial_s - i\partial_t$, A = 0th-order

$\Rightarrow$ If D is $\mathbb{C}$-linear, local existence thm $\Rightarrow$ $\exists!$ antihol. VB str. on E s.t. $\eta \in \Gamma(E)$ antihol. iff $D\eta = 0$.

EX: If $E, F \to \dot{\Sigma}$ have antihol. VB strs., then $\mathrm{Hom}_{\mathbb{C}}(E,F)$ inherits an antihol. str. ($\Rightarrow$ also an anti-CR op.) s.t. $D(\Phi\eta) = (D\Phi)\eta + \Phi D\eta$

$\forall\ \eta \in \Gamma(E)$, $\Phi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(E,F))$.

ex 1: If D is a CR-op., then $-D^* : \Gamma(F) \to \Gamma(\underset{\substack{\cong \\ \mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F)}}{E})$ is anti-CR.

ex 2: If D is a CR-op., $\exists$ an anti-CR op. $\bar{D} : \Gamma(\bar{E}) \to \Gamma(\underset{\substack{\cong \\ \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)}}{\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, \bar{E})})$

$\bar{D}\bar{\eta} := \overline{D\eta}$ (recall canonical $\mathbb{C}$-antilin. iso. $E \to \bar{E} : \eta \mapsto \bar{\eta}$, identity map)

<u>prop</u>: Given $D_r = D + rB$ for D a CR-op., $B \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(E,F))$,

$\exists$ $B_1 \in \Gamma(\mathrm{End}_{\mathbb{R}}(E))$ s.t. $\forall$ $r \in \mathbb{R}$, $\eta \in \Gamma(E)$

$$\boxed{D_r^* D_r \eta = D^* D\eta + \underbrace{\underset{\text{"positive"}}{r^2 B^* B\eta} + r B_1 \eta}_{\text{0th-order}}}$$

<u>Pf</u>: Write $B\eta = \beta\bar{\eta}$ for some $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E},F))$.

(1) Assume D is $\mathbb{C}$-linear: then $\bar{D}$ & $-D^*$ defn. antihol. VB strs. on $\bar{E}$ & F $\Rightarrow$ they determine an antihol VB str. & anti-CR op. ∂_H on $\mathrm{Hom}_{\mathbb{C}}(\bar{E},F)$.

$$D_r^* D_r \eta = (D^* + rB^*)(D + rB)\eta = D^*D\eta + r^2 B^*B\eta + r[B^*D\eta + D^*(B\eta)]$$

$$B^*D\eta + D^*(B\eta) = B^*D\eta - (-D^*(\beta\bar{\eta})) = B^*D\eta - ((\partial_H\beta)\bar{\eta} + \beta\bar{D}\bar{\eta})$$

$$= -(\partial_H\beta)\bar{\eta} + (B^* - B)D\eta$$

<u>EX</u>: In local trivs., B, B^* look like $\mathbb{C}^{n\times n}$-valued fns. s.t. $B^* = B^T$

so in case $n=1$, $B^* - B \equiv 0$.

$$\Rightarrow D_r^* D_r \eta = D^*D\eta + r^2 B^*B\eta - r(\partial_H\beta)\bar{\eta}$$

(2) D not $\mathbb{C}$-linear: then $D = D_{\mathbb{C}} + A$ where $D_{\mathbb{C}}\eta := \frac{1}{2}(D\eta - JD(J\eta))$ is also a CR-op., $A : E \to F$ antilin, i.e. $A\eta = \alpha\bar{\eta}$ for some $\alpha \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E},F))$. $D_r = D_{\mathbb{C}} + (A + rB)$

$$\Rightarrow (D^*D - D_{\mathbb{C}}^* D_{\mathbb{C}})\eta = A^*A\eta - (\partial_H\alpha)\bar{\eta}$$

$$(D_r^* D_r - D_{\mathbb{C}}^* D_{\mathbb{C}})\eta = (A + rB)^*(A + rB)\eta - (\partial_H\alpha + r\,\partial_H\beta)\bar{\eta}$$

$$D_r^* D_r \eta - D^*D\eta = r^2 B^*B\eta + r\underbrace{[(A^*B + B^*A)\eta - (\partial_H\beta)\bar{\eta}]}_{B_1\eta}$$

▨

pf of index formula for $\Gamma = \emptyset$, $n = 1$ & $\chi(\Sigma) + 2c_1(E) = 0$

To show: $\operatorname{ind} D = 0$. Choose $B \in \Gamma(\underbrace{\mathrm{Hom}_{\mathbb{C}}(E,F)}_{c_1(\cdot)=0})$ to be nowhere zero,

then $|B\eta| \geq c|\eta|$ for some const $c > 0$.

claim: $D_r := D + rB$ is injective $\forall\ r \gg 0$.

pf: $\|D_r \eta\|^2_{L^2} = \langle D_r \eta, D_r \eta\rangle_{L^2} = \langle \eta, D_r^* D_r \eta\rangle_{L^2}$

$$= \underbrace{\langle \eta, D^* D\eta\rangle_{L^2}}_{\|D\eta\|^2_{L^2} \geq 0} + r^2 \underbrace{\langle \eta, B^* B\eta\rangle_{L^2}}_{\|B\eta\|^2_{L^2} \geq c^2 \|\eta\|^2_{L^2}} + r \underbrace{\langle \eta, B_1 \eta\rangle_{L^2}}_{\leq c_1 \|\eta\|^2_{L^2}}$$

$$\geq r^2 c^2 \|\eta\|^2_{L^2} - r c_1 \|\eta\|^2_{L^2} = r^2 \underbrace{\left(c^2 - \frac{c_1}{r}\right)}_{>0 \text{ if } r \gg 0} \|\eta\|^2_{L^2} > 0 \text{ if } \eta \neq 0$$

for $r \gg 0$.

For $-D_r^* = -D^* - rB^*$ is a family of CR-ops on F $\Rightarrow$ by same arg., D_r^* also inj. $\forall\ r \gg 0$.

$\Rightarrow \forall\ r \gg 0$, D_r is an iso, $\Rightarrow \operatorname{ind}(D) = \operatorname{ind}(D_r) = 0$. ☒