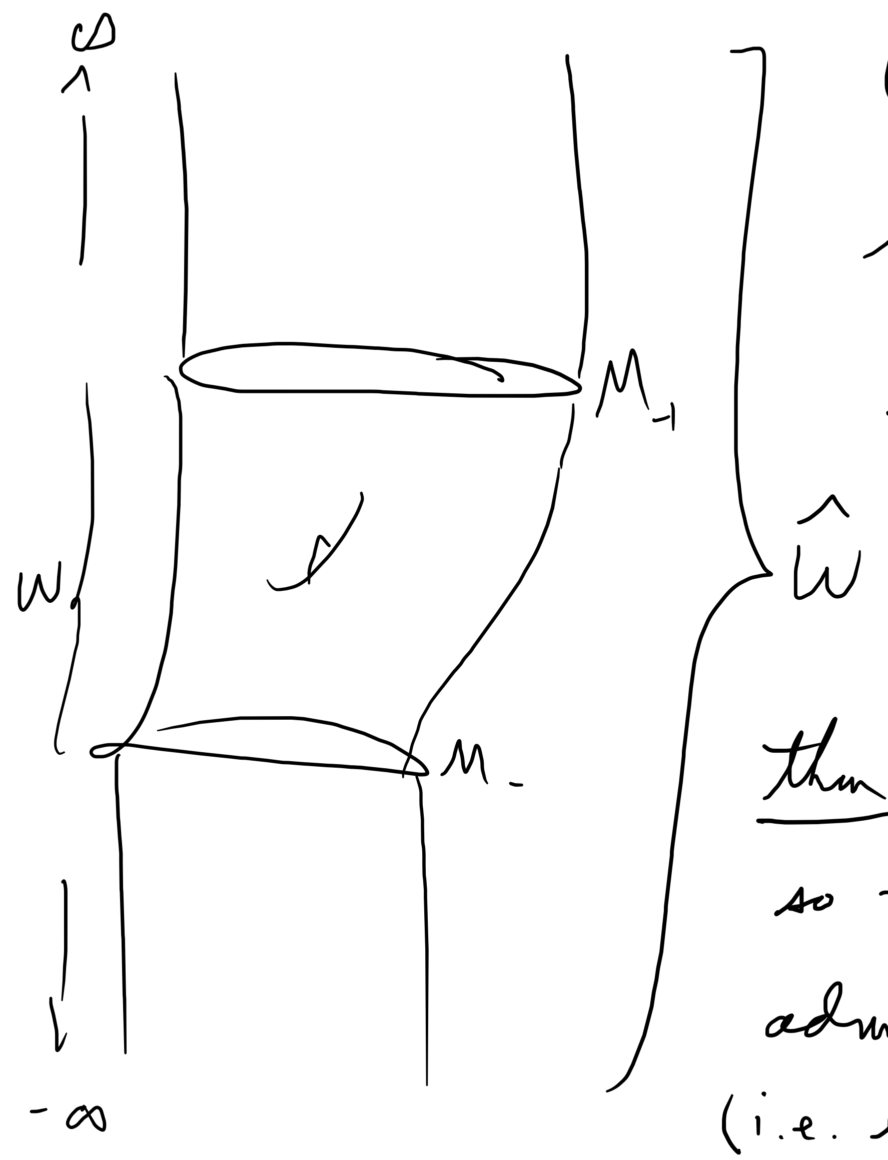
$\partial(W^{2n},\omega) = -M_- \amalg M_+$ sympl. cobordism w/ stable bdry $\xrightarrow{\text{completion}}$ $\widehat{W}$, $J \in \mathcal{J}(\omega, \mathcal{H}_+, \mathcal{H}_-)$

(w/ SHS's $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$) $\quad \gamma^\pm := (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$ nondeg closed Reeb orbits in $M_\pm$



⇝ moduli space of unparametrized J-hol. curves

$$\mathcal{M}(J) := \mathcal{M}_{g,m}(J, A, \gamma^+, \gamma^-) = \{(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)\}/\sim$$

A: rel. hom. class in $H_2(W, \bar{\gamma}^+ \cup \bar{\gamma}^-)$. $\quad \Sigma$: genus g; Γ^+: k_+; Γ^-: k_-; Θ: m

<u>thm 1</u> ("IFT"): $\exists$ an open subset $\mathcal{M}^{reg}(J) \subseteq \mathcal{M}(J)$ of so-called Fredholm regular curves in $\mathcal{M}(J)$, s.t. $\mathcal{M}^{reg}(J)$ admits the structure of a fin.-dim. smooth <u>orbifold</u> (i.e. locally $\cong \mathbb{R}^N/$action of a finite "local isotropy group")

of dim. $= \boxed{(n-3)(2-2g-k_+-k_-) + 2c_1^\tau(A) + \sum_{i=1}^{k_+} \mu_{CZ}^\tau(\gamma_i^+) - \sum_{i=1}^{k_-} \mu_{CZ}^\tau(\gamma_i^-) + 2m}$

the map $ev : \mathcal{M}^{reg}(J) \longrightarrow \widehat{W}^{\times m}$ is smooth,

& the local isotropy group at $u \in \mathcal{M}^{reg}(J)$ is $\mathrm{Aut}(u)$.

<u>rk</u>: vir-dim $\mathcal{M}(J) :=$ the number on the RHS. $\quad$ <u>case</u> $m=0$: $\mathrm{ind}(u)$ for $u \in \mathcal{M}(J)$

"virtual dimension" $\quad$ "index of u"

<u>cor</u>: $\{u \in \mathcal{M}^{reg}(J) \mid \mathrm{Aut}(u) = \{\mathrm{Id}\}\}$ is a mfd of dim $=$ vir-dim.

<u>notation</u>: $\tau :=$ choice of unitary trivs. for $(\gamma_i^\pm)^*\xi_\pm$,

$c_1^\tau(A) := c_1^{\tau'}(u^*T\widehat{W})$ where $\tau' := \mathrm{Id} \oplus \tau$ on $T(\mathbb{R}\times M_\pm) = \epsilon_\pm \oplus \xi_\pm$, $\quad \epsilon_\pm := \mathrm{Span}\{\partial_r, R_\pm\}$

<u>EX</u>: (1) $c_1^\tau(A)$ deps only on τ & A

(2) $2c_1^\tau(A)$ + CZ-terms is indep. of τ.

<u>analytic setup</u>: $k \in \mathbb{N}$, $1 < p < \infty$ s.t. $kp > 2$ $(W^{k,p} \hookrightarrow C^0)$, $\delta \geq 0$.

Fix $\dot{\Sigma} = (\Sigma \setminus (\Gamma^+ \cup \Gamma^-))$, $\Theta = \{\zeta_1, \ldots, \zeta_m\} \subseteq \dot{\Sigma}$, Σ has genus g,

fix cpx str. j_Γ near Γ & hol. cyl. coords $(s,t) \in Z_\pm$ near each $z_i^\pm \in \Gamma^\pm$.

For an asymp. Herm. VB $E \to \dot{\Sigma}$, let

$$W^{k,p,\delta}(E) := \left\{ \eta \in W^{k,p}_{loc}(E) \;\middle|\; \text{in cyl. coord \& asymp. triv. near } z_i^\pm \in \Gamma^\pm,\ \|e^{\pm\delta s}\eta\|_{W^{k,p}(Z_\pm)} < \infty \right\}.$$

$$\mathcal{B} := \mathcal{B}^{k,p,\delta} := \left\{ \exp_f h : \dot{\Sigma} \to \widehat{W} \;\middle|\; \begin{array}{l} f \in C^\infty(\dot{\Sigma}, \widehat{W}) \text{ \& near } z_j^\pm \in \Gamma^\pm, \\ f(s,t) = (T_j^\pm s + a, \gamma_j^\pm(t+b)) \\ \text{for } T_j^\pm > 0 \text{ the period of } \gamma_j^\pm : S^1 \to M_\pm, \\ a \in \mathbb{R},\ b \in S^1 \text{ arbitrary constants} \\ h \in W^{k,p,\delta}(f^* T\widehat{W}) \end{array} \right\}.$$

<u>Lemma</u> (cf. Eliasson, "Geom. of mfds of maps", JDG 1967):

$\mathcal{B}$ has a natural smooth Banach mfd structure s.t. $ev : \mathcal{B} \to \widehat{W}^{\times m}$, $u \mapsto (u(\zeta_1), \ldots, u(\zeta_m))$

is C^∞, & $\forall\, u \in \mathcal{B}$,

$T_u \mathcal{B} = W^{k,p,\delta}(u^* T\widehat{W}) \oplus V_\Gamma$ for some space of smooth sections

$V_\Gamma \subseteq \Gamma(u^* T\widehat{W})$ which are constants lin. combinations of ∂_r & $R_\pm$ near

each puncture $(\Rightarrow \dim V_\Gamma = 2(k_+ + k_-))$. $\square$

nonlinear CR-op.: For a cpx str. j on $\dot{\Sigma}$ matching j_Γ near Γ,

$\bar\partial_{j,J} : \mathcal{B} \longrightarrow \mathcal{E} : u \longmapsto du + J(u) \circ du \circ j$ is a smooth section of a Banach space bndl $\mathcal{E}$ with fibers $\mathcal{E}_u = W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^* T\hat{W}))$.

why smooth? In local coords., $\bar\partial_{j,J}(u) = \partial_s u + J(u) \partial_t u \in W^{k-1,p,\delta}$

$u \longmapsto \partial_s u : W^{k,p} \to W^{k-1,p}$ contin. linear $\Rightarrow C^\infty$.

$\longmapsto \partial_t u$

$u \longmapsto J(u) = J \circ u : W^{k,p} \longrightarrow W^{k,p}$ is of class C^r if $J \in C^{k+r}$, $r \in \mathbb{N}$.

We're assuming $\boxed{J \in C^\infty}$!

$(J(u), \partial_t u) \longmapsto J(u)\partial_t u : W^{k,p} \times W^{k-1,p} \longrightarrow W^{k-1,p}$ contin. bilinear (assuming $kp > 2$).

$\Rightarrow C^\infty$.

linearization: $D_u := D\bar\partial_{j,J}(u) : T_u\mathcal{B} \longrightarrow \mathcal{E}_u$ is a linear CR-type op.,

$T_u\mathcal{B} = W^{k,p,\delta} \oplus V_\Gamma$, $\mathcal{E}_u = W^{k-1,p,\delta}$

is Fredholm iff its restriction $W^{k,p,\delta} \xrightarrow{D_\delta} W^{k-1,p,\delta}$ is Fredholm,

$\mathrm{ind}\, D_u = \mathrm{ind}\, D_\delta + \dim V_\Gamma = \mathrm{ind}\, D_\delta + 2(\#\Gamma)$.

recall: If $u = u_\gamma$ a trivial cylinder, then $(D_u \eta)\partial_s = \partial_s - \begin{pmatrix} -i\partial_t & \\ & A_\gamma \end{pmatrix}$

on $u_\gamma^* T(\mathbb{R} \times M) = u_\gamma^* \epsilon \oplus u_\gamma^* \xi$.

cor: D_u is C^∞-asymptotic to the asymp. op. $(-i\partial_t) \oplus A_{\gamma_j^\pm}$ at $z_j^\pm \in \Gamma^\pm$.

(asymp. op. on $T(\mathbb{R} \times M_\pm) = \epsilon_\pm \oplus \xi_\pm$ along $\gamma_j^\pm$).

trouble: $-i\partial_t$ is a degenerate asymp. op.

prop 1: D_δ is Fredholm if $\delta > 0$ is suff. small, &

$$\operatorname{ind} D_\delta = n\chi(\dot\Sigma) + 2c_1^\tau(u^*T\hat W) - \#\Gamma + \sum_{i=1}^{k_+} \mu_{CZ}^\tau(\gamma_i^+) - \sum_{i=1}^{k_-} \mu_{CZ}^\tau(\gamma_i^-).$$

pf:
$$\begin{tikzcd} W^{k,p} \arrow[r, "\eta \mapsto e^f \eta", "\cong"'] \arrow[d, "D_\delta'"] & W^{k,p,\delta} \arrow[d, "D_\delta"] \\ W^{k-1,p} \arrow[r, "\eta \mapsto e^f \eta", "\cong"'] & W^{k-1,p,\delta} \end{tikzcd}$$

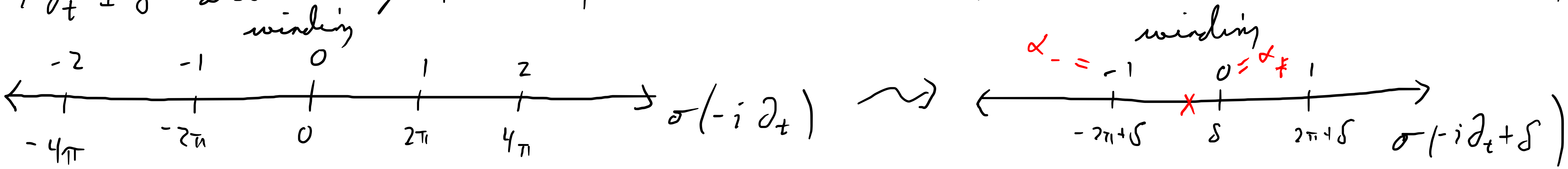
for a fn. $f \in C^\infty(\dot\Sigma, \mathbb{R})$ s.t. $f(s,t) = \mp \delta s$ near $\Gamma^\pm$.

D_δ' is also a CR-type op.

C^∞-asymp. to $(-i\partial_t \pm \delta) \oplus (A_{\gamma_j^\pm} \pm \delta)$ at $z_j^\pm \in \Gamma^\pm$.

$A_{\gamma_j^\pm}$ nondeg $\Rightarrow$ $A_{\gamma_j^\pm} \pm \delta$ still nondeg if $0 < \delta \ll 1$ & CZ-index unchanged.

$-i\partial_t \pm \delta$ also nondeg if $\delta \ll 1$: recall $\mu_{CZ}(A) = 2\alpha_-(A) + p(A) = 2\alpha_+(A) - p(A)$



$p = \alpha_+ - \alpha_- = 1$

$\Rightarrow \mu_{CZ}(-i\partial_t + \delta) = 2\alpha_+ - p = -1$

sim. $\mu_{CZ}(-i\partial_t + \delta) = 2\alpha_- + p = +1$.

$\Rightarrow \mu_{CZ}^\tau((-i\partial_t \pm \delta) \oplus (A_{\gamma_j^\pm} \pm \delta)) = \mu_{CZ}^\tau(\gamma_j^\pm) \mp 1$.

Apply formula from 2 weeks ago. ∎

varying j on Σ: Let $\mathcal{J}(\Sigma) := \{$cpx strs. on Σ compatible w/ given orientation$\}$.

$\Theta \subseteq \Sigma$ finite subset.

$\mathrm{Diff}(\Sigma, \Theta) := \{ \varphi : \Sigma \xrightarrow{\text{diffeo}} \Sigma \mid$ or.-pres. & $\varphi|_\Theta = \mathrm{Id} \}$.

$\cup\!\!|$

$\mathrm{Diff}_0(\Sigma, \Theta)$ cpnt containing Id. Both act on $\mathcal{J}(\Sigma)$ by $\varphi \cdot j := \varphi^* j$.

$\rightsquigarrow \mathcal{M}_{g,m} := \mathcal{J}(\Sigma) / \mathrm{Diff}(\Sigma, \Theta)$ "moduli space of Riemann surfaces of genus g w/ m marked pts", $m = \#\Theta$

$\mathcal{T}(\Sigma, \Theta) := \mathcal{J}(\Sigma) / \mathrm{Diff}_0(\Sigma, \Theta)$ Teichmüller space.

$\mathrm{Aut}(\Sigma, j, \Theta) := \{ \varphi : (\Sigma, j) \to (\Sigma, j) \mid$ bihol. & $\varphi|_\Theta = \mathrm{Id} \} =$ stabilizer of j under $\mathrm{Diff}(\Sigma, \Theta)$-action

prop 2: (1) $\mathcal{T}(\Sigma, \Theta)$ is a smooth mfd & $\forall\, j \in \mathcal{J}(\Sigma)$, $\mathrm{Aut}(\Sigma, j, \Theta)$ is a Lie group w/ $\boxed{\dim \mathrm{Aut}(\Sigma, j, \Theta) - \dim \mathcal{T}(\Sigma, \Theta) = 3\chi(\Sigma) - 2m}$.

(2) If $\chi(\Sigma \setminus \Theta) < 0$ (i.e. (Σ, j, Θ) is stable), then $\mathrm{Aut}(\Sigma, j, \Theta)$ is discrete, so $\dim \mathcal{T}(\Sigma, \Theta) = -3\chi(\Sigma) + 2m = 6g - 6 + 2m$.

pf sketch: Consider $\bar\partial_j : \mathcal{B}_\Theta \longrightarrow \mathcal{E} : \varphi \longmapsto d\varphi + j \circ d\varphi \circ j$ where

$\mathcal{B}_\Theta := \{ \varphi \in W^{k,p}(\Sigma, \Sigma) \mid \varphi|_\Theta = \mathrm{Id} \}$, $\quad \mathcal{E}_\varphi := W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, \varphi^* T\Sigma))$.

$\bar\partial_j^{-1}(0) = \mathrm{Aut}(\Sigma, j, \Theta) \Rightarrow T_{\mathrm{Id}} \mathrm{Aut}(\Sigma, j, \Theta) \cong \ker\left(D_{\mathrm{Id}} := D\bar\partial_j(\mathrm{Id}) : T_{\mathrm{Id}} \mathcal{B}_\Theta \to \mathcal{E}_{\mathrm{Id}}\right)$

$T_{\mathrm{Id}} \mathcal{B}_\Theta = \{ X \in W^{k,p}(T\Sigma) \mid X|_\Theta = 0 \} \overset{\text{codim} = 2m}{\subseteq} W^{k,p}(T\Sigma)$.

$D_{\mathrm{Id}} : W^{k,p} \to W^{k-1,p}$ is Fredholm w/ index $= \chi(\Sigma) + 2c_1(T\Sigma) = 3\chi(\Sigma)$

$\Rightarrow \mathrm{ind}(D_{\mathrm{Id}} : T_{\mathrm{Id}} \mathcal{B}_\Theta \to \mathcal{E}_{\mathrm{Id}}) = 3\chi(\Sigma) - 2m$.

claim: $\mathrm{im}\, D_{\mathrm{Id}} = T_j(\mathrm{Diff}_0(\Sigma, \Theta) \cdot j)$, hence $T_{[j]} \mathcal{T}(\Sigma, \Theta) = \mathrm{coker}\, D_{\mathrm{Id}}$

pf: $\varphi_\rho \in \mathrm{Diff}(\Sigma, \Theta)$ 1-param. fam. s.t. $\varphi_0 = \mathrm{Id}$, $\partial_\rho \varphi_\rho|_{\rho=0} =: X \in T_{\mathrm{Id}} \mathcal{B}_\Theta$.

$$\partial_\rho (\varphi_\rho^* j)|_{\rho=0} = \partial_\rho \left(T\varphi_\rho^{-1} \circ j \circ T\varphi_\rho \right)\Big|_{\rho=0} = j \circ \nabla X - \nabla X \circ j$$
$$= j\,(\nabla X + j \circ \nabla X \circ j) = j\, D_{\mathrm{Id}} X = D_{\mathrm{Id}}(jX).$$

$\square$