Σ genus g, closed, $\Theta \subseteq \Sigma$ $m \geq 0$ pts $\rightsquigarrow \mathcal{M}_{g,m} := \mathcal{J}(\Sigma) / \mathrm{Diff}(\Sigma,\Theta)$

<u>defn</u>: (Σ, j, Θ) is <u>stable</u> if $\chi(\Sigma \setminus \Theta) < 0$. $\mathcal{T}(\Sigma,\Theta) := \mathcal{J}(\Sigma) / \mathrm{Diff}_0(\Sigma,\Theta)$

<u>prop</u>: If $\chi(\Sigma \setminus \Theta) < 0$, then $\mathrm{Diff}_0(\Sigma,\Theta)$ acts freely & properly on $\mathcal{J}(\Sigma)$, & $\mathrm{Aut}(\Sigma, j, \Theta)$ is finite $\forall j$.

(<u>rk</u>: $\mathrm{Diff}(\Sigma,\Theta)$ also acts properly, w/ finite stabilizers $\cong \mathrm{Aut}(\Sigma, j, \Theta)$.)

<u>sketch of pf</u>: <u>free</u>: $\varphi \in \mathrm{Diff}_0(\Sigma,\Theta)$, $\varphi \neq \mathrm{Id}$, $\varphi^* j = j$ for some $j \in \mathcal{J}(\Sigma)$.

$\Rightarrow \varphi : (\Sigma, j) \to (\Sigma, j)$ is bihol. $\Rightarrow$ its fixed pts are isolated & positive

Lefschetz $\Rightarrow$ $0 \leq \# \mathrm{Fix}(\varphi) = \chi(\Sigma)$ since $\varphi \overset{h}{\sim} \mathrm{Id}$. But $\varphi|_\Theta = \mathrm{Id}$ $\Rightarrow$

$\# \mathrm{Fix}(\varphi) = \chi(\Sigma) \geq m \Rightarrow \chi(\Sigma \setminus \Theta) \geq 0$ contra!

(<u>note</u>: Can also show $\ker(D_{\mathrm{Id}} : T_{\mathrm{Id}} \mathcal{B}_\Theta \to \mathcal{E}_{\mathrm{Id}}) = \{0\}$ in stable case.

$\Rightarrow \nexists$ elements of $\mathrm{Aut}(\Sigma, j, \Theta)$ close to Id, i.e. group is discrete.)

<u>proper</u>: To show: $\mathrm{Diff}_0(\Sigma,\Theta) \times \mathcal{J}(\Sigma) \to \mathcal{J}(\Sigma) \times \mathcal{J}(\Sigma) : (\varphi, j) \mapsto (\varphi^* j, j)$

is proper, i.e. if $\varphi_n \in \mathrm{Diff}_0(\Sigma,\Theta)$ & $j_n \in \mathcal{J}(\Sigma)$ are seqs. s.t.

$j_n \to j$ & $\varphi_n^* j_n =: j_n' \to j'$, φ_n has a C^∞-conv. subseq.

$\varphi_n : (\Sigma, j_n') \xrightarrow{\deg 1} (\Sigma, j_n)$ is a seq of hol maps w/ "bdd energy", i.e.

for any area form $d\mathrm{vol}$ on Σ, $E(\varphi_n) := \int_\Sigma \varphi_n^* d\mathrm{vol} = \mathrm{vol}(\Sigma)$ bdd

If either $g \geq 1$ or $\#\Theta \geq 3$, then $\nexists$ "bubbling" $\Rightarrow \varphi_n$ is compact.

<u>cor</u>: $\mathrm{Aut}(\Sigma, j, \Theta)$ is cpt $\Rightarrow$ also finite. $\square$

Teichmüller slices: Fix $[(\Sigma, j_0, \Gamma^+, \Gamma^-; \Theta, u_0)] \in \mathcal{M}(J) := \mathcal{M}_{g,m}(J, A, \gamma^+, \gamma^-)$,

let $\Theta' := \Gamma \cup \Theta$, $G_0 := \mathrm{Aut}(\Sigma, j_0, \Theta')$.

Lemma: $\exists$ a smoothly embedded fin.-dim. family of cpx strs. $\widehat{\mathcal{T}} \subseteq \mathcal{J}(\Sigma)$ s.t.

(i) $j_0 \in \widehat{\mathcal{T}}$, (ii) all $j \in \widehat{\mathcal{T}}$ matches j_0 near Γ,

(iii) $\widehat{\mathcal{T}}$ is G_0-inv't (iv) $T_{j_0}\widehat{\mathcal{T}}$ is a closed complement of

$$\operatorname{im}\left(D_{\mathrm{Id}} : T_{\mathrm{Id}} \mathcal{B}_{\Theta'} \to \mathcal{E}_{\mathrm{Id}}\right) \subseteq \Gamma\left(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, j_0)\right)$$

Recall: $\operatorname{coker} D_{\mathrm{Id}} \cong T_{[j_0]} \mathcal{T}(\Sigma, \Theta')$

Cor: $\widehat{\mathcal{T}} \to \mathcal{T}(\Sigma, \Theta') : j \mapsto [j]$ defines a homeo. $\mathrm{nbhd}(j_0) \xrightarrow{\cong} \mathrm{nbhd}([j_0])$.

Pf of Lemma (if $\chi(\Sigma \setminus \Theta') < 0$):

Choose a G_0-inv't L^2-pairing on $\Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, j_0))$, let

$V := L^2$-ortho compl. of $\operatorname{im} D_{\mathrm{Id}} = \ker D_{\mathrm{Id}}^* \subseteq \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, j_0))$.

After an L^r-small change of V (multiplying by cutoff fns. vanishing near Γ),

can assume $y = 0$ near Γ $\forall y \in V$. Fix $\varepsilon > 0$ small,

set $\widehat{\mathcal{T}} := \left\{ \left(\mathrm{Id} + \frac{1}{2} j_0 y\right) j_0 \left(\mathrm{Id} + \frac{1}{2} j_0 y\right)^{-1} \,\middle|\, y \in V \text{ s.t. } \|y\|_{C^0} < \varepsilon \right\}$. ∎

Extend $\mathcal{E} \longrightarrow \mathcal{B}$ to a Banach space bdl over $\mathcal{T} \times \mathcal{B}$ s.t.

$\mathcal{E}_{(j,u)} = W^{k-1,p,\delta}\left(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\dot{\Sigma}, j), (u^* T\hat{W}, J))\right)$, a defn C^∞-section

$\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \longrightarrow \mathcal{E} : (j,u) \longmapsto \bar{\partial}_{j,J}(u) = du + J(u) \circ du \circ j$. $(j_0, u_0) \in \bar{\partial}_J^{-1}(0)$

since $u_0 \in \mathcal{B}^{k,p,\delta}$ (asymptotic regularity) if $\delta > 0$ suff. small.

$D\bar{\partial}_J(j,u) : T_j \mathcal{T} \oplus T_u \mathcal{B} \longrightarrow \mathcal{E}_{(j,u)} : (y, \eta) \longmapsto D_u \eta + J(u) \circ du \circ y$.

<u>defn</u>: $u_0 \in \mathcal{M}(J)$ is <u>Fredholm regular</u> if $D\bar{\partial}_J(j_0,u_0)$ is surjective.

<u>observe</u>: $\delta > 0$ small $\Rightarrow$ $D\bar{\partial}_J(j_0,u_0)$ is Fredholm w/ index

$= \dim \mathcal{T} + \operatorname{ind} D_u = \dim G_0 - \left(3\chi(\Sigma) - 2(\#\Gamma + m)\right) + \dim V_\Gamma + \operatorname{ind} D_\delta$

$= \dim G_0 - 3\chi(\Sigma) + \underline{2(\#\Gamma)} + 2m + \underline{2(\#\Gamma)} + n\chi(\dot{\Sigma}) + 2c_1^\tau(u^* T\hat{W}) + CZ\text{-terms} - \underline{\#\Gamma}$

$= \dim G_0 + \underbrace{(n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(A) + CZ\text{-terms} + 2m}_{= \text{vir-dim}\, \mathcal{M}(J)}$

$\overset{(IFT)}{\Longrightarrow}$ <u>Lemma</u>: If u_0 is Fredholm reg., $\bar{\partial}_J^{-1}(0) \subseteq \mathcal{T} \times \mathcal{B}$ near (j_0,u_0) is a smooth mfd of dim $= \dim G_0 + \text{vir-dim}\, \mathcal{M}(J)$. $\square$

G_0 acts smoothly on $\bar\partial_J^{-1}(0)$ by $\varphi\cdot(j,u) := (\varphi^* j, u\circ\varphi)$,

w/ finite stabilizer $= \mathrm{Aut}(u) \Rightarrow \bar\partial_J^{-1}(0)/G_0$ is a smooth orbifold of dim $=$ vir. dim.

<u>Lemma</u>: $\bar\partial_J^{-1}(0)/G_0 \longrightarrow \mathcal{M}(J) : [(j,u)] \longmapsto [(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)]$

is a homeo $\mathrm{nbhd}([(j_0,u_0)]) \longrightarrow \mathrm{nbhd}([(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)])$.

<u>pf sketch</u>: To show it is surj. onto a nbhd:

Given $[(\Sigma, j_n, \Gamma^+, \Gamma^-, \Theta, u_n)] \longrightarrow [(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)]$, i.e.

$j_n \xrightarrow{C^\infty(\Sigma)} j_0$, $\quad u_n \xrightarrow{C^\infty_{loc}(\dot\Sigma)} u_0$, $\quad \bar u_n \xrightarrow{C^0(\bar\Sigma)} \bar u_0$, need to show these are in the image for n large. <u>Trouble</u>: $j_n \in \mathcal{J}(\Sigma)$ but $j_n \notin \mathcal{T}$ in general.

$[j_n] \longrightarrow [j_0] \in \mathcal{T}(\Sigma, \Theta') \Rightarrow \exists\ \varphi_n \in \mathrm{Diff}_0(\Sigma, \Theta')$ s.t. $j_n' := \varphi_n^* j_n \in \mathcal{T}$

& $j_n' \longrightarrow j_0$. $\quad \varphi_n : (\Sigma, j_n') \longrightarrow (\Sigma, j_n)$, by properness, $\exists$ conv. subseq.

$\varphi_n \xrightarrow{C^\infty} \varphi_\infty \in \mathrm{Aut}(\Sigma, j_0, \Theta') \cap \mathrm{Diff}_0(\Sigma, \Theta') \xRightarrow{\text{freeness}} \varphi_\infty = \mathrm{Id}$.

Now replace j_n with j_n', u_n with $u_n' := u_n \circ \varphi_n \longrightarrow u_0$ as above.

$(j_n', u_n') \in \bar\partial_J^{-1}(0)$ for n large. Asymp. reg. $\Rightarrow u_n' \xrightarrow{W^{k,p,\delta}} u_0$

$\Rightarrow (j_n', u_n') \longrightarrow (j_0, u_0)$ in $\bar\partial_J^{-1}(0)$. $\square$

Proof of Thm 1 is complete.

transversality: $E \to B$ a V.B., $s: B \to E$ a smooth section, call $x \in s^{-1}(0)$ regular if $Ds(x): T_xB \to E_x$ is surjective ($\iff$ $s(B) \subseteq E$ intersects the 0-section transversely ($\pitchfork$) at x.)

goal: Prove for "generic" $s \in \Gamma(E)$, all $x \in s^{-1}(0)$ are regular.

idea: Choose a "suff. large" space of perturbations $X \overset{\text{submfd}}{\subseteq} \Gamma(E)$, consider a "universal moduli space" $\mathcal{U} := \{(s,x) \in X \times B \mid s(x) = 0\} = f^{-1}(0)$

for a section $f: X \times B \to E' : (s,x) \mapsto s(x)$, $E' :=$ pullback E along proj. $X \times B \to B$.

$Df(s,x)(t,v) = Ds(x)v + t(x) \Rightarrow$ if X is suff. large, the map

$T_sX \to E'_{(s,x)} : t \mapsto t(x)$ is surj. $\Rightarrow Df(s,x)$ is surj (hard part)

$\overset{IFT}{\Longrightarrow} \mathcal{U}$ is a mfd.

standard part: Consider projection $\mathcal{U} \xrightarrow{\pi} X : (s,x) \mapsto s$,

$d\pi(s,x) : \ker Df(s,x) \to T_sX : (t,v) \mapsto t$.

linear algebra EX: If $Df(s,x)$ is surj., then $\exists$ natural isos.

$\ker d\pi(s,x) \cong \ker Ds(x)$, $\quad$ $\operatorname{coker} d\pi(s,x) \cong \operatorname{coker} Ds(x)$.

$\Rightarrow$ If $Ds(x)$ is Fredholm (+ surj.), then $d\pi(s,x)$ is also Fredholm (+ surj)

Sard-Smale thm (Smale 1965): X, Y separable Banach mfds,

$f: X \to Y$ a map of class C^k $(k \geq 1)$ s.t.

(i) $\forall x \in X$, $df(x): T_xX \to T_{f(x)}Y$ is Fredholm, (ii) $k \geq \operatorname{ind} df(x) + 1$.

Then $Y^{\text{reg}} := \{y \in Y \mid df(x) \text{ is surj. } \forall x \in f^{-1}(y)\}$ is a comeager subset of Y.

(comeager := contains a ctbl int. of open-dense sets $\overset{\text{Baire}}{\Longrightarrow}$ dense.)

Apply to $\pi: \mathcal{U} \to X \Rightarrow \exists$ comeager $X^{\text{reg}} \subseteq X$ s.t. $\forall s \in X^{\text{reg}}$,

all $x \in s^{-1}(0)$ are regular.