transversality (in cobordisms)

Fix $(W,\omega) \rightsquigarrow$ completion $\widehat{W}$, $\partial W = (-M_-, \mathcal{H}_-) \amalg (M_+, \mathcal{H}_+)$

$\mathcal{U} \overset{\text{open}}{\subseteq} W$ w/ cpt closure, $J^{\text{fix}} \in \mathcal{J}(\omega, \mathcal{H}_+, \mathcal{H}_-)$.

$\mathcal{J}_{\mathcal{U}} := \{ J \in \mathcal{J}(\omega, \mathcal{H}_+, \mathcal{H}_-) \mid J = J^{\text{fix}}$ outside $\mathcal{U}\}$ is a complete metrizable space w.r.t. C^∞-top.

thm: $\exists$ a comeager subset $\mathcal{J}_{\mathcal{U}}^{\text{reg}} \subseteq \mathcal{J}_{\mathcal{U}}$ s.t. $\forall J \in \mathcal{J}_{\mathcal{U}}^{\text{reg}}$, every curve in

$\mathcal{M}^*(J) := \{ u \in \mathcal{M}(J) \mid u$ has an injective pt. w/ image $\mathcal{U}\}$

is Fredholm regular. cor: For generic J, $\mathcal{M}^*(J)$ is a mfd of dim = vir. dim $\mathcal{M}(J)$.

(recall: $u : \dot{\Sigma} \longrightarrow \widehat{W}$ has $z \in \dot{\Sigma}$ as an injective pt. if $du(z) \neq 0$ & $u^{-1}(u(z)) = \{z\}$.

Condition in $\mathcal{M}^*(J)$ means u is simple & $u(\dot{\Sigma}) \cap \mathcal{U} \neq \emptyset$.)

step 1: Universal moduli space

Needed: a "suff. large" Banach mfd of a.c.s.'s in which to vary J.

$\mathcal{J}_{\mathcal{U}}$ is not a Banach mfd (C^∞-top.)

idea A: a.c.s.'s of class C^m $(m < \infty)$: $\mathcal{J}_{\mathcal{U}}^m$ is a Banach mfd.

Choose $J^{\text{ref}} \in \mathcal{J}_{\mathcal{U}}$. Think of $\mathcal{J}_{\mathcal{U}}$ as a Fréchet mfd w/ tangent space

$$T_{J^{\text{ref}}} \mathcal{J}_{\mathcal{U}} = \left\{ Y \in \Gamma\left(\overline{\operatorname{End}}_{\mathbb{C}}(T\widehat{W}, J^{\text{ref}})\right) \;\middle|\; \begin{array}{l} Y \equiv 0 \text{ outside } \mathcal{U}, \\ \omega(Y\cdot, J^{\text{ref}}\cdot) + \omega(J^{\text{ref}}\cdot, Y\cdot) = 0 \end{array} \right\}$$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ Y \longmapsto \left(\mathbb{1} + \tfrac{1}{2} J^{\text{ref}} Y\right) J^{\text{ref}} \left(\mathbb{1} + \tfrac{1}{2} J^{\text{ref}} Y\right)^{-1} =: \text{"}\exp_{J^{\text{ref}}} Y\text{"} \end{array}$$

C^0-nbhd of $0 \xrightarrow{\cong} C^0$-nbhd of J^{ref} in $\mathcal{J}_{\mathcal{U}}$.

Similarly, $\mathcal{J}_{\mathcal{U}}^m = \{ \exp_{J^{\text{ref}}} Y \mid J^{\text{ref}} \in \mathcal{J}_{\mathcal{U}},\ Y$ as above but of class $C^m\}$.

Trouble: if $J \in C^m \setminus C^\infty$, $\bar{\partial}_J$ is no longer smooth $\Rightarrow \mathcal{M}^{\text{reg}}(J)$ is not

a smooth orbifold.

<u>idea B</u>: "<u>Floer</u> <u>C_ε-space</u>"

Fix $J^{ref} \in \mathcal{J}_\mathcal{U}$, $\varepsilon := (\varepsilon_\ell > 0)_{\ell=0}^\infty$ s.t. $\varepsilon_\ell \to 0$, $c > 0$ small,

<u>lemma</u>: $\{ Y \in T_{J^{ref}} \mathcal{J}_\mathcal{U} \mid \|Y\|_{C_\varepsilon} := \sum_{\ell=0}^\infty \varepsilon_\ell \|Y\|_{C^\ell} < \infty \}$ is a separable Banach space w.r.t. $\|\cdot\|_{C_\varepsilon}$,

& if ε_ℓ conv. to 0 suff. fast, then $\forall$ $x \in \mathcal{U}$, this space contains sections w/ arbitrary values at x & arbitrarily small support around x. $\boxed{}$

$\rightsquigarrow$ Banach mfd (w/ one chart): $\mathcal{J}_\mathcal{U}^\varepsilon := \{ J = \exp_{J^{ref}} Y \mid \|Y\|_{C_\varepsilon} < \infty,\ \|Y\|_{C^0} < c,\ Y \in T_{J^{ref}} \mathcal{J}_\mathcal{U} \}$.

<u>obvious</u> <u>lemma</u>: Inclusion $\mathcal{J}_\mathcal{U}^\varepsilon \hookrightarrow \mathcal{J}_\mathcal{U}$ is continuous.

C^∞-topology $\boxed{}$

<u>defn</u>: $\mathcal{M}^*(\mathcal{J}_\mathcal{U}^\varepsilon) := \{ (u, J) \mid J \in \mathcal{J}_\mathcal{U}^\varepsilon,\ u \in \mathcal{M}^*(J) \}$.

<u>goal</u>: $\mathcal{M}^*(\mathcal{J}_\mathcal{U}^\varepsilon)$ is a Banach mfd.

Nbhd of $([(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)], J_0)$ in $\mathcal{M}^*(\mathcal{J}_u^\varepsilon)$

$\cong$ nbhd of $[(j_0, u_0, J_0)]$ in $\bar\partial^{-1}(0)/G$ where

$\bar\partial : \mathcal{T} \times \mathcal{B}^{k,p,\delta} \times \mathcal{J}_u^\varepsilon \longrightarrow \mathcal{E} : (j, u, J) \longmapsto du + J(u) \circ du \circ j \in W^{k-1,p,\delta}(\overline{\mathrm{Hom}} \cdots)$

$G := \mathrm{Aut}(\Sigma, j_0, \Gamma \cup \Theta)$ acting by $\varphi \cdot (j, u, J) := (\varphi^* j, u \circ \varphi, J)$

acts properly & freely (since all u simple $\Rightarrow \mathrm{Aut}(u) = \{\mathrm{Id}\}$).

$$D\bar\partial(j_0, u_0, J_0) : T_{j_0}\mathcal{T} \oplus \underbrace{T_{u_0}\mathcal{B}}_{W^{k,p,\delta}(u_0^* T\hat{W}) \oplus V_\Gamma} \oplus T_{J_0}\mathcal{J}_u^\varepsilon \longrightarrow \mathcal{E}_{(j_0, u_0, J_0)}$$

$$(y, \eta, Y) \longmapsto \overbrace{\underbrace{D_{u_0}\eta + Y(u_0) \circ du_0 \circ j_0}_{=: L(\eta, Y)} + J_0(u_0) \circ du_0 \circ y}^{D\bar\partial_J(j_0, u_0)}$$

main lemma: $L : W^{k,p,\delta}(u_0^* T\hat{W}) \oplus T_{J_0}\mathcal{J}_u^\varepsilon \longrightarrow W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot\Sigma, u_0^* T\hat{W}))$

is surjective.

pf for $k = 1$ (general case follows by elliptic reg.)

$L : W^{1,p,\delta} \oplus T_{J_0}\mathcal{J}_u^\varepsilon \longrightarrow L^{p,\delta} : (\eta, Y) \longmapsto D_{u_0}\eta + Y(u_0) \circ du_0 \circ j_0$.

D_{u_0} is Fredholm $\overset{(EX)}{\Longrightarrow}$ im L is closed, & L has a bdd right inverse if surj.

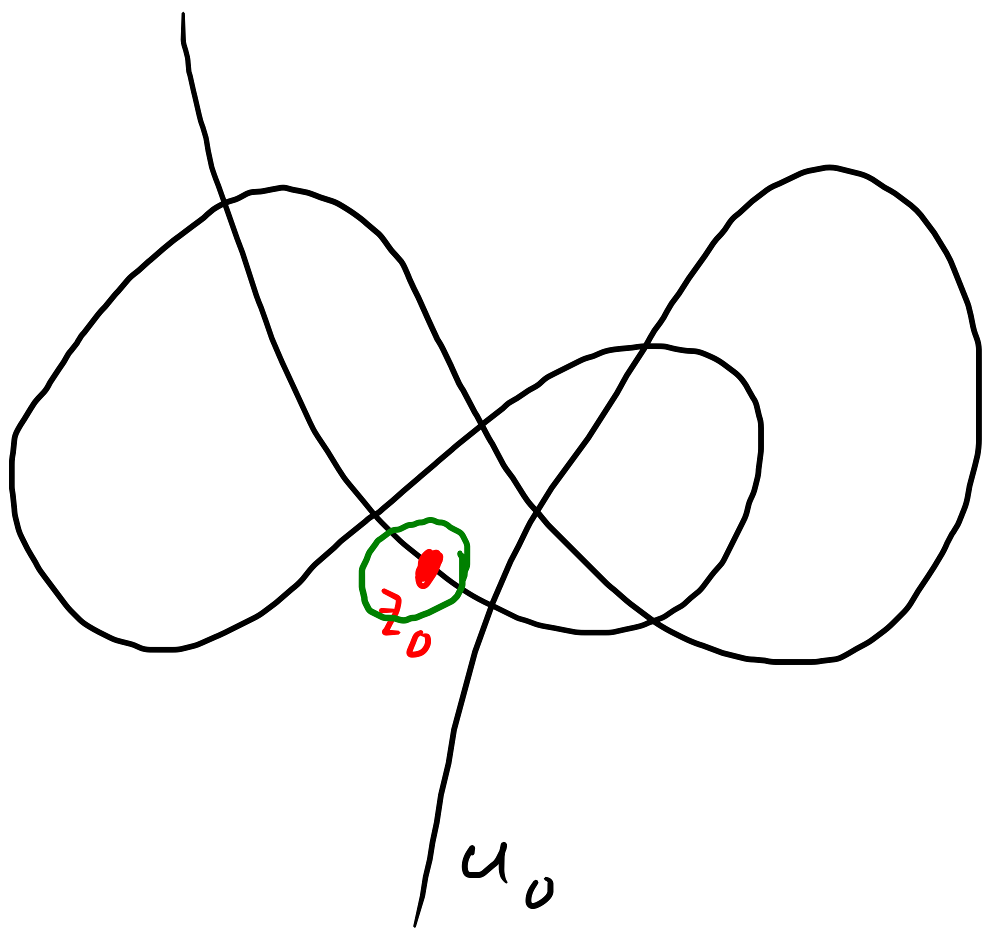
If L not surj., Hahn-Banach $\Rightarrow$ $\exists$ $\theta \neq 0 \in (L^{p,\delta})^* \cong L^{q,-\delta}$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

s.t. $\langle L(\eta, Y), \theta \rangle_{L^2} = 0$ $\forall\ \eta, Y$ $\Rightarrow$ (1) $\langle D_{u_0}\eta, \theta\rangle_{L^2} = 0$ $\forall\ \eta \in W^{1,p,\delta}$

(2) $\langle Y(u_0)\circ du_0 \circ j_0, \theta\rangle_{L^2} = 0$ $\forall\ Y \in T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}}$.

(1) $\Rightarrow$ $D^*_{u_0}\theta = 0$ $\xRightarrow{\text{(reg.)}}$ $\theta \in C^\infty$ & (similarity princ.) θ has only isolated zeroes.

Choose an inj. pt. $z_0 \in \dot{\Sigma}$ s.t. $u_0(z_0) \in \mathcal{U}$ & WLOG $\theta(z_0) \neq 0$.



$du_0(z_0) \neq 0$ $\Rightarrow$ can pick $Y \in T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}}$ supported near

$u_0(z_0)$ s.t. $Y(u_0)\circ du_0 \circ j_0 = \theta$ at z_0

$\Rightarrow$ $\langle Y(u_0)\circ du_0 \circ j_0, \theta\rangle_{L^2} > 0$ (works only because z_0 is an inj. pt.)

Contra! $\blacksquare$

<u>step 2</u> : <u>Sard-Smale thm</u>

Main lemma $\Rightarrow$ $\mathcal{M}^*(\mathcal{J}^\varepsilon_u)$ is a smooth separable Banach mfd,

proj. $\mathcal{M}^*(\mathcal{J}^\varepsilon_u) \xrightarrow{\pi} \mathcal{J}^\varepsilon_u : (u, J) \longmapsto J$ is C^∞

$D\bar\partial_J(j, u)$ is Fredholm $\Rightarrow$ $d\pi(u, J)$ also Fredholm & is surj. iff $u \in \mathcal{M}(J)$ is Fredholm reg.

Sard-Smale $\Rightarrow$

<u>cor</u> : $\exists$ comeager subset $\mathcal{J}^{\varepsilon, reg}_u \subseteq \mathcal{J}^\varepsilon_u$ s.t. $\forall$ $J \in \mathcal{J}^{\varepsilon, reg}_u$,
every $u \in \mathcal{M}^*(J)$ is Fredholm regular. ▨

J^{ref} was chosen arbitrarily $\Rightarrow$
space of $J \in \mathcal{J}_u$ s.t. all $u \in \mathcal{M}^*(J)$ are regular is <u>dense</u> (in C^∞).

<u>step 3</u> : <u>C_ε to C^∞</u> : Let $\mathcal{J}^{reg}_u := \{ J \in \mathcal{J}_u \mid \text{all } u \in \mathcal{M}^*(J) \text{ are regular} \}$.

Previous cor. $\Rightarrow$ $\mathcal{J}^{reg}_u$ is dense.

<u>lemma 1</u> : $\forall$ $J \in \mathcal{J}_u$, $\exists$ nested seq. $\mathcal{M}^*_1(J) \subseteq \mathcal{M}^*_2(J) \subseteq \dots \subseteq \mathcal{M}^*(J)$ s.t.

(1) $\bigcup_{N \in \mathbb{N}} \mathcal{M}^*_N(J) = \mathcal{M}^*(J)$

(2) $\forall$ cpt $K \subseteq \mathcal{J}_u$, $\{ (u, J) \mid J \in K,\ u \in \mathcal{M}_N(J) \}$ is cpt. $\forall N$ ▨

Now let $\mathcal{J}^N_u := \{ J \in \mathcal{J}_u \mid \text{all } u \in \mathcal{M}^*_N(J) \text{ are regular} \}$. this is dense in $\mathcal{J}_u$, but also open due to (2) in Lemma 1.

$\mathcal{J}^{reg}_u = \bigcap_{N \in \mathbb{N}} \mathcal{J}^N_u$ is therefore comeager. ▨

extension: $\{J_s \in \mathcal{J}_\mathcal{U}\}_{s\in P}$, P = smooth fin. dim. mfd,

$\mathcal{M}(\{J_s\}) := \{(u,s) \mid s \in P,\ u \in \mathcal{M}(J_s)\}$. "parametric moduli space"

locally $\cong \bar\partial_{\{J_s\}}^{-1}(0)/G$ for $\bar\partial_{\{J_s\}} : \mathcal{T} \times \mathcal{B} \times P \longrightarrow \mathcal{E} : (j,u,s) \mapsto \bar\partial_{j,J_s}(u)$.

Call $(u,s) \in \mathcal{M}(\{J_s\})$ parametrically regular if $D\bar\partial_{\{J_s\}}(j,u,s)$ is surj.

$\Rightarrow$ open subset $\mathcal{M}^{reg}(\{J_s\}) \subseteq \mathcal{M}(\{J_s\})$, orbifold of dim = vir-dim $\mathcal{M}(J_s)$ + dim P.

EX: $u \in \mathcal{M}(J_s)$ is regular $\iff$ (u,s) is parametrically regular & is a regular pt. of $\mathcal{M}(\{J_s\}) \longrightarrow P : (u,s) \mapsto s$.

thm: For $V \overset{open}{\subseteq} P$ w/ cpt closure, any family $\{J_s^{fix}\}_{s\in P}$,

for generic families $\{J_s \in \mathcal{J}_\mathcal{U}\}_{s\in P}$ matching J_s^{fix} outside $\mathcal{U}$ & everywhere for $s \in P \setminus V$, all $(u,s) \in \mathcal{M}(\{J_s\})$ s.t. $s \in V$ & u has an inj pt. mapped into $\mathcal{U}$ are parametrically regular.

ex: $P = [0,1]$, $V = (0,1)$, i.e. generic homotopies in $\mathcal{J}_\mathcal{U}$ w/ fixed end pts.

vir-dim $\mathcal{M}(J_s) = 0$

J_0 & J_1 generic.

$\exists$ parameter values $s \in (0,1)$ s.t. J_s not generic