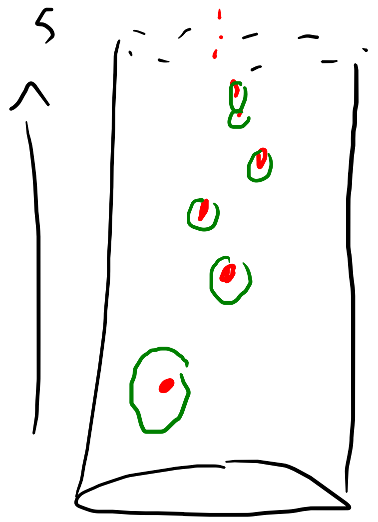
quick pf of removal of singularities (C^0-extension):

$u : Z_+ \longrightarrow \hat{W}$ J-hol. w/ J tamed by a symp. form Ω, $\int_{Z_+} u^*\Omega < \infty$,

$\overline{u(Z_+)} \overset{\text{cpt}}{\subseteq} \hat{W}$.

(1) $|du|$ bdd on Z_+: else $\exists$ seq. of reparams.

$v_k(z) := u\left(z_k + \frac{z}{R_k}\right)$ $\quad (z \in \mathbb{D}_{\underbrace{\epsilon_k R_k}_{\to\infty}} \subseteq \mathbb{C})$ "bubble off" to a



nonconstant $v_\infty : \mathbb{C} \longrightarrow \hat{W}$ J-hol. w/ $\int_{\mathbb{C}} v_\infty^*\Omega = 0$ contra!

(2) $\forall$ $s_k \longrightarrow \infty$, $u_k(s,t) := u(s+s_k, t)$ on $[-s_k, \infty) \times S^1$

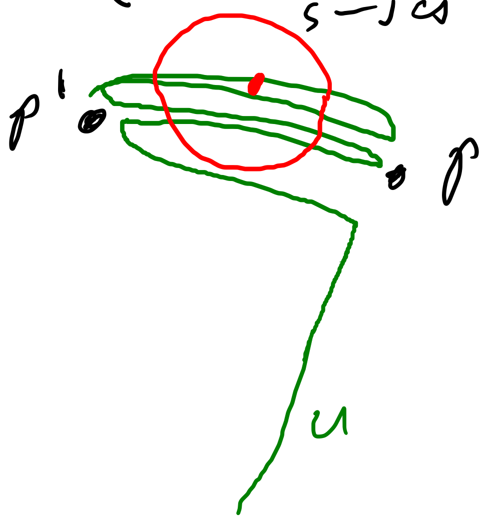
are C^1-bdd $\Longrightarrow$ subseq. $\xrightarrow{C^\infty_{loc}}$ $u_\infty : \mathbb{R} \times S^1 \longrightarrow \hat{W}$ J-hol.

w/ $\int_{\mathbb{R}\times S^1} u_\infty^*\Omega = 0$ $\Longrightarrow$ u_∞ = const.

$\Longrightarrow$ $\forall$ $s_k \longrightarrow \infty$, $u(s_k, \cdot)$ has a subseq. $C^\infty(S^1)$-conv. to a const. $p \in \hat{W}$.

(3) If $s_k, s_k' \longrightarrow \infty$ s.t. $u(s_k, \cdot) \longrightarrow p$, $u(s_k', \cdot) \longrightarrow p'$, claim: $p = p'$.

($\Longrightarrow$ $\lim_{s\to\infty} u(s, \cdot) = $ const) pf by monotonicity lemma: $\blacksquare$



$\Longrightarrow$ $\int_{Z_+} u^*\Omega = \infty$. similar monotonicity arg. also $\Longrightarrow$

(1) $u : (\dot{\Sigma}, j) \longrightarrow (\hat{W}, J)$ has $E(u) < \infty$ & no removable ptys $\Longrightarrow$ u is a proper map.

(2) If $u(s_k, \cdot) \xrightarrow{C^\infty(S^1)} \gamma$ an isolated closed orbit, then

dist$(u(s,\cdot)$, reparametrization of $\gamma) \xrightarrow{s\to\infty} 0$.

Assume $[(\Sigma, j_k, \Gamma^+, \Gamma^-, \Theta, u_k)] \in \mathcal{M}(J_k)$ for some C^∞-conv. seq. $J_k \to J$ in $\mathcal{J}(\omega, \mathcal{H}_+, \mathcal{H}_-)$

& $E(u_k)$ bdd.

$\underline{Q}$: If u_k has no conv. subseq. & no limit in $\mathcal{M}(J)$, what can happen?

(1) $\underline{\text{bubbling}}$: Spse $j_k \xrightarrow{C^\infty} j$, u_k are C^0-bdd on cpct subsets,

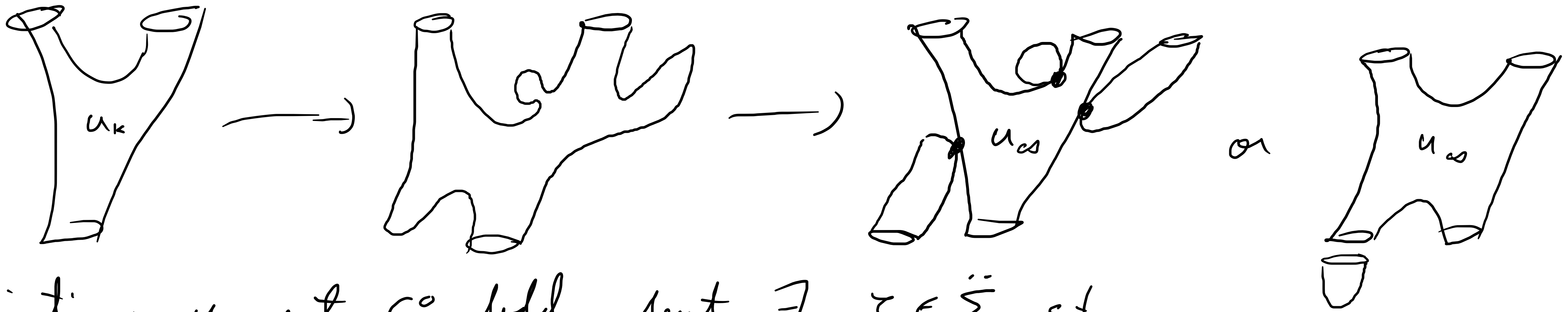
but $\exists$ seqs $\{z_k^{(i)} \in \dot{\Sigma}\}_{k=1}^\infty$ for $i = 1, \dots, N$ s.t. $z_k^{(i)} \to z^{(i)} \in \dot{\Sigma}$

all disjoint, s.t. $|du_k(z_k^{(i)})| \to \infty$ $\forall i$ but $|du|$ bdd on cpct subsets of $\dot{\Sigma} \setminus \bigcup_{i=1}^N \{z^{(i)}\}$.

Then $\exists$ subseq. $u_k \xrightarrow{C^\infty_{loc}} u_\infty : \ddot{\Sigma} := \dot{\Sigma} \setminus \bigcup_{i=1}^N \{z^{(i)}\} \to \hat{W}$.

Also $\exists$ rescaled subseqs $u_k \circ \varphi_k^{(i)} : \mathbb{D}_{\epsilon_k R_k} \to \hat{W}$, $\xrightarrow{C^\infty_{loc}} u_\infty^{(i)} : \mathbb{C} \to \hat{W}$

all with $E < \infty$. Either sing. at $\infty \in S^2$ is removable & $u_\infty^{(i)}$ extends to $S^2 \to \hat{W}$ ("bubble"), or $u_\infty^{(i)}$ is asymp. cyl.



$\underline{\text{variation}}$: u_k not C^0-bdd, but $\exists$ $\zeta \in \ddot{\Sigma}$ s.t.

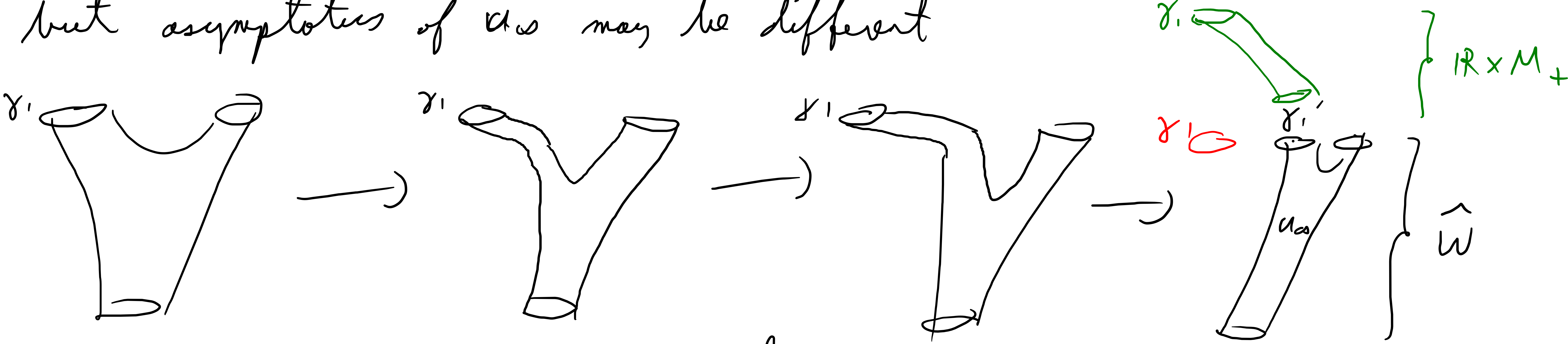
$u_k(\zeta) \in \{r_k\} \times M_+ \subseteq \hat{W}$ for $r_k \to \infty$, then after $\mathbb{R}$-translation,

can make u_k a C^1-bdd seq. of maps into $\mathbb{R} \times M_+$,

$\Rightarrow$ $u_\infty : \ddot{\Sigma} \to \mathbb{R} \times M_+$.

breaking: Assume again $j_k \xrightarrow{C^\infty} j$, also $u_k : \dot{\Sigma} \to \hat{W}$ C^1-bdd on cpt subsets $\Rightarrow$ subseq. $u_k \xrightarrow{C^\infty_{loc}(\dot{\Sigma})} u_\infty : \dot{\Sigma} \to \hat{W}$ w/ $E(u_\infty) < \infty$

but asymptotics of u_∞ may be different



Take an end $Z_+ \subseteq \dot{\Sigma}$ & reparametrize as

$v_k(s,t) := u_k(s + s_k, t)$ for some $s_k \to \infty$. After $\mathbb{R}$-translation,

can arrange for v_k to be C^1-bdd into $\mathbb{R} \times M_+$,

$\Rightarrow$ subseq. $v_k \to v_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M_+$

degenerating cpx str.: Maybe $j_k \in \mathcal{J}(\Sigma)$ has no conv. subseq.

$\mathcal{M}_{g,\ell} := \{(\Sigma, j, \underbrace{\Gamma^+ \cup \Gamma^- \cup \Theta}_{\Theta'})\}/\sim$ is noncpt in general ⇝ Deligne–Mumford compactification

$\ell = k_+ + k_- + m$

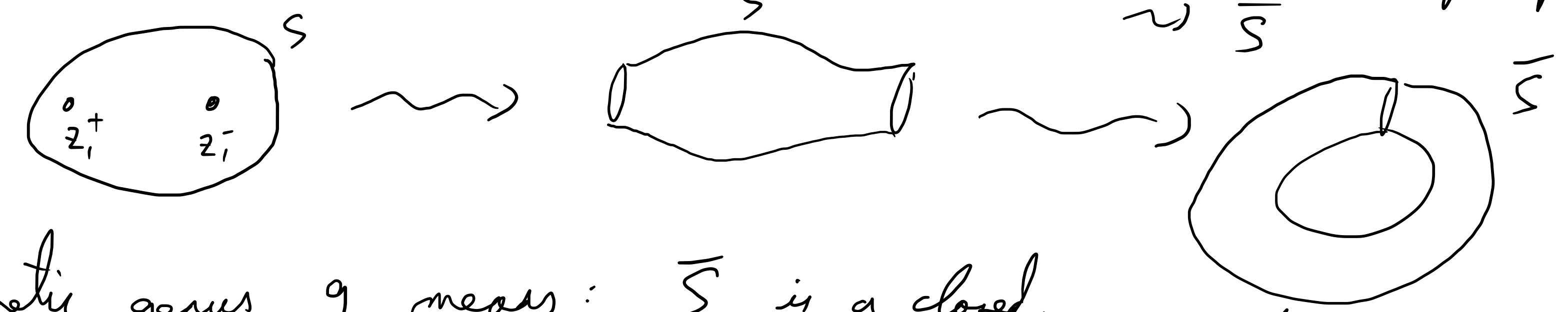
$\overline{\mathcal{M}}_{g,\ell} := \{(S, j, \Theta', \Delta)\}/\sim$

S: maybe disconnected, but closed

$\Delta = \{\{z_1^+, z_1^-\}, \dots, \{z_N^+, z_N^-\}\}$ "nodes"

$S \setminus \Delta \cong$ Riem. surf. w/ cyl. ends $\overset{\text{circle compactification}}{\rightsquigarrow} \hat{S}$

For each $\{z_i^+, z_i^-\} \in \Delta$, glue together corresp. cpts of $\partial \hat{S}$ ⇝ $\overline{S}$



arithmetic genus g means: $\overline{S}$ is a closed, conn. surface of genus g.

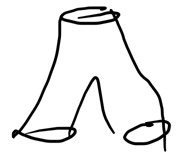
$(\Sigma, j, \Theta') \in \mathcal{M}_{g,\ell}$ is stable iff $\chi(\Sigma \setminus \Theta') < 0 \iff 2g + \ell \geq 3$.

(S, j, Θ', Δ) is stable if all cpts of $S \setminus (\Theta' \cup \Delta)$ have $\chi < 0$.

$\overline{\mathcal{M}}_{g,\ell}$ consists only of stable nodal Riem. surfs.

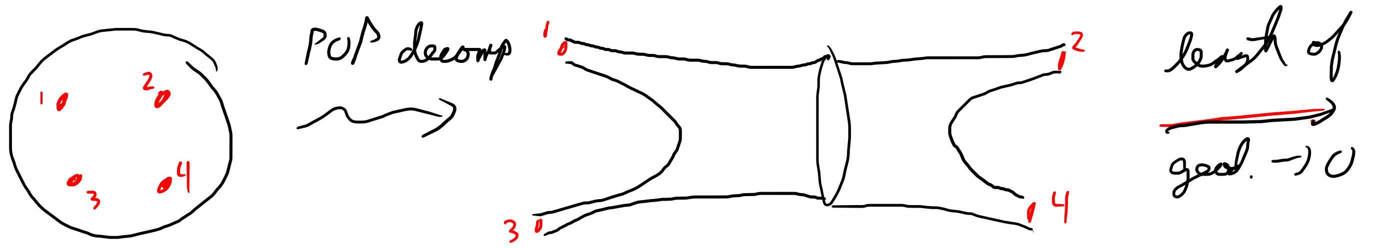
lemma (hyperbolic geometry): If $2g + \ell \geq 3$, then $\Sigma \setminus \Theta'$ admits a ! complete Riem. metric compatible w/ j s.t. curvature $\equiv -1$.

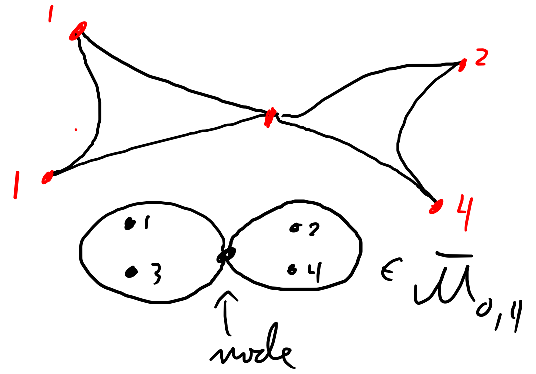
$\exists$ a pair-of-pants decomposition splitting $\Sigma \setminus \Theta'$ into surfaces $\cong$



along closed geodesics of universally bdd length.

ex: $\mathcal{M}_{0,4} \cong \{(S^2, i, (0, 1, \infty, z_4)) \mid z_4 \in S^2 \setminus \{0, 1, \infty\}\} \cong S^2 \setminus \{0, 1, \infty\}$





$\exists$ 2 other possibilities

thm: Every seq. in $\mathcal{M}_{g,\ell}$ $(2g + \ell \geq 3)$ has a subseq. conv. in $\overline{\mathcal{M}}_{g,\ell}$.

__defn of $\overline{\mathcal{M}}_{g,m}(A,J,\gamma^+,\gamma^-)$__ (SFT - compactification)

Consists of "stable hol. buildings of height $N_+|1|N_-$ ($\forall$ integers $N_\pm \geq 0$) w/ arithmetic genus g & m marked pts., up to biholomorphic equivalence

$\overline{\mathcal{M}}_{g,m}(A,J,\gamma^+,\gamma^-) \ni [(S,j,\Gamma^+,\Gamma^-,\Theta,\Delta^{nd},\Delta^{br},L,\Phi,u)]$

closed Riem. surf. (maybe discon.) — punct. — marked pts — nodes (= unordered sets of unordered pairs in $S\setminus(\Gamma\cup\Theta)$)

__level structure__: $L : S \longrightarrow \{-N_-,\dots,-1,0,1,\dots,N_+\}$ locally const. s.t.

$\forall\ \{z^+,z^-\} \in \Delta^{nd}$ ("nodes"), $\quad L(z^+) = L(z^-)$

$\forall\ \{z^+,z^-\} \in \Delta^{br}$ ("breaking pts"), $\quad L(z^+) = L(z^-)+1$

__decoration__ Φ: for each $\{z^+,z^-\}\in\Delta^{br}$, Φ defines an orientation-reversing diffeo between corresp. cpnt of $\partial\hat{S}$

$u : (\dot{S} := S\setminus(\Gamma\cup\Delta^{br}), j) \longrightarrow \coprod_{-N_-\leq N\leq N_+} (\hat{W}_N, J_N)$ asymp. cyl.

where $\hat{W}_N := \hat{W}$ for $N=0$, $\quad \mathbb{R}\times M_\pm$ for $\pm N > 0$.

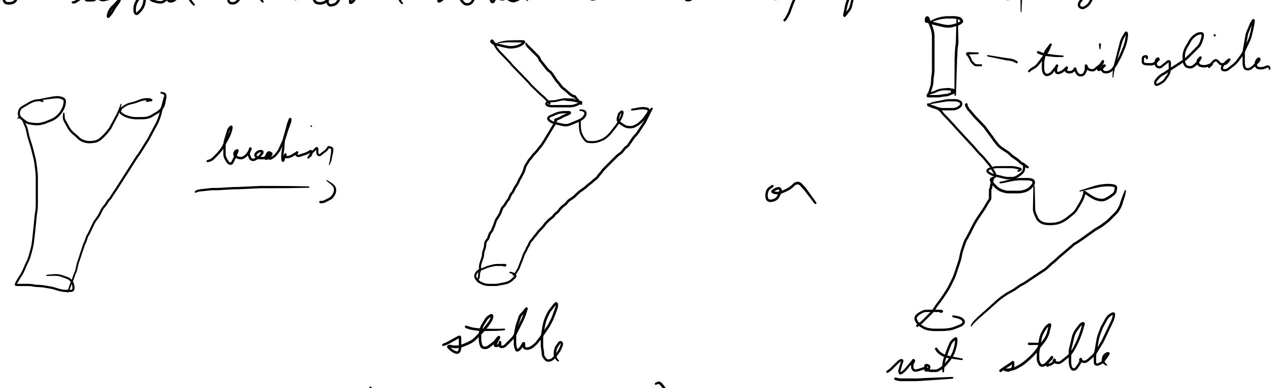
$\overline{S} :=$ cpct surf. w. bndy obtained from $\hat{S}$ by gluing together bndy cpnts for nodal pairs in Δ^{nd} & Δ^{br} (using Φ).

($\exists$ bndy cpnts due to $\Gamma^\pm$).

Require that u has a contin. extension to a map def'd on $\overline{S}$ (const. on circles arising from Δ^{nd})

Call the building __stable__ if on every cpnt of $S\setminus(\Gamma\cup\Theta\cup\Delta^{nd}\cup\Delta^{br})$ where $\chi\geq 0$, $u \neq$ const. ($\Rightarrow$ building has a finite automorphism group)

+ no upper or lower level consists only of trivial cylinders without nodes.



__SFT cptness thm__ (in cobordism): Every seq. of asymp cyl. J-hol. curves w/ bdd energy in $\hat{W}$ has a subseq conv. to (only one!) stable hol. building

In particular, $\overline{\mathcal{M}}_{g,m}(A,J,\gamma^+,\gamma^-)$ is a cpct Hausdorff space. ∎