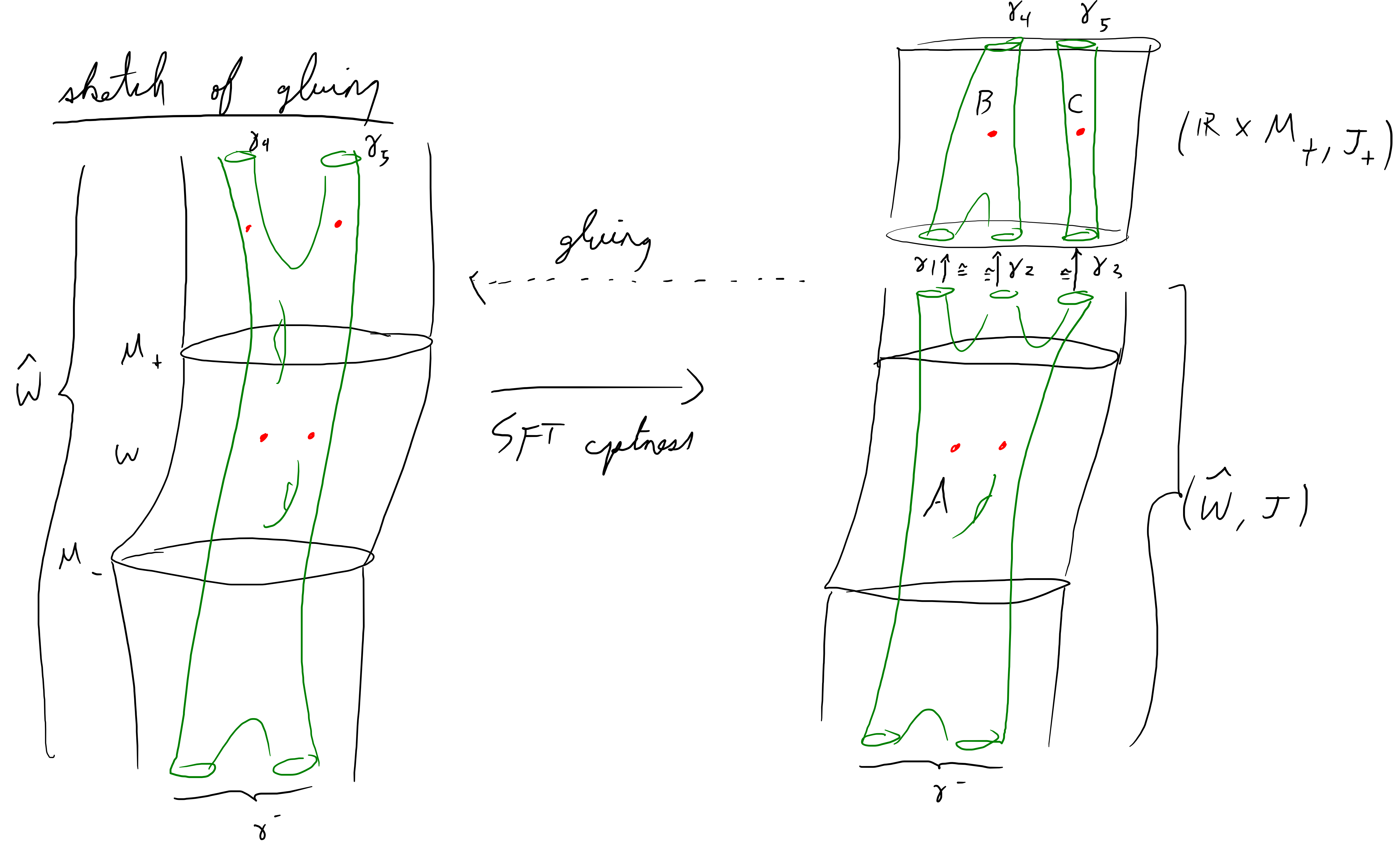
sketch of gluing



Fix a pt. p_γ on the image of each orbit γ.

Given $[(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)] \in \mathcal{M}_{g,m}(J, A, \gamma^+, \gamma^-) =: \mathcal{M}(J)$, an asymptotic marker at $z \in \Gamma^\pm$ (asymp. to orbit γ_z) is a choice of ray $\ell \subseteq T_z\Sigma$ s.t. $\lim\limits_{\substack{w \to z \\ \text{along } \ell}} u(w) = (\pm\infty, p_{\gamma_z})$.

A choice of asymp. marker at each cpunct in a hol. building determines a decoration at each breaking orbit.

If $\mathrm{cov}(\gamma_z) = \kappa_z$, then $\exists$ κ_z choices of asymp. marker at z.

Let $\mathcal{M}^\$(J) = \mathcal{M}^\$_{g,m}(J, A, \gamma^+, \gamma^-) := \left\{ \begin{array}{l} \text{curves in } \mathcal{M}_{g,m}(J,A,\gamma^+,\gamma^-) \text{ endowed w/} \\ \text{asymp. markers at every pt.} \end{array} \right\}$

<u>Lemma</u>: If $\#\Gamma \geq 1$, then the set of reg. curves in $\mathcal{M}^\$(J)$ is a <u>mfd</u> (not orbifold) w/ $\dim \mathcal{M}^\$(J) = \dim \mathcal{M}(J)$.

<u>Pf</u>: Recall $\mathcal{M}(J) \overset{\text{loc.}}{\cong} \bar{\partial}_J^{-1}(0) / G$ for $\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \longrightarrow \mathcal{E}$ nonlinear CR-op.

$[(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)] \in \mathcal{M}(J)$, $\quad [(j_0, u_0)] \in \bar{\partial}_J^{-1}(0)/G$, $\quad G := \mathrm{Aut}(\Sigma, j_0, \Gamma \cup \Theta)$.

$\forall\ (j,u) \in \bar{\partial}_J^{-1}(0)$, $\exists\ \underbrace{\prod_{z \in \Gamma} \kappa_z}_{=: \kappa \in \mathbb{N}}$ choices of asymp. markers

$\rightsquigarrow$ κ-fold covering space $\widetilde{\mathcal{M}} \longrightarrow \bar{\partial}_J^{-1}(0)$ s.t. $\mathcal{M}^\$(J) \overset{\text{loc}}{\cong} \widetilde{\mathcal{M}}/G$.

G acts <u>freely</u> on $\widetilde{\mathcal{M}}$: acts on nbhds of each pt. by roots of unity, $\Rightarrow$ fixes a marker iff $= \mathrm{Id}$ near that pt. $\overset{\text{unique contin.}}{\Longleftrightarrow} \equiv \mathrm{Id}$ globally.

$\Rightarrow \widetilde{\mathcal{M}}/G$ is a mfd. $\square$

Let $\mathcal{M}_A := \mathcal{M}^{\$}_{1,2}(J, A, (\gamma_1, \gamma_2, \gamma_3), \gamma^-)$

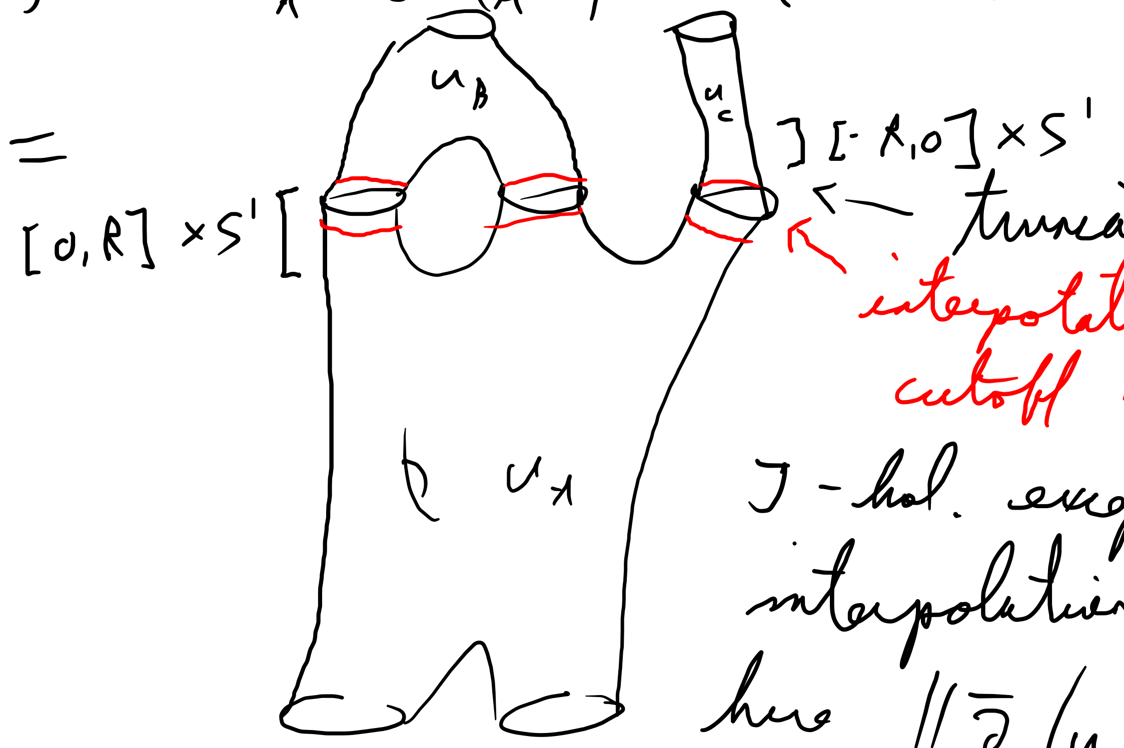
$\mathcal{M}_B := \mathcal{M}^{\$}_{0,1}(J_+, B, \gamma_4, (\gamma_1, \gamma_2))$ $\quad$ $\mathcal{M}_C := \mathcal{M}^{\$}_{0,1}(J_+, C, \gamma_5, \gamma_3)$.

$\hat{\mathcal{M}}_{BC} \subseteq \mathcal{M}_B \times \mathcal{M}_C$ a hypersurface $\pitchfork$ $\mathbb{R}$-action.

(then $\hat{\mathcal{M}}_{BC} \cong (\mathcal{M}_B \times \mathcal{M}_C)/\mathbb{R}$).

<u>pregluing map</u>: For $u_A \in \mathcal{M}_A$, $(u_B, u_C) \in \mathcal{M}_{BC}$, $R \geq 0$ large

$u_A \#_R (u_B, u_C) :=$



truncate!

interpolation using cutoff fn.

J-hol. except in interpolation region, here $\|\bar\partial_J(u_A \#_R(u_B, u_C))\|$ small for $R >> 0$.

pos. end

W

neg. end

"quantitative IFT" $\Rightarrow$

$\forall R >> 0$, $\exists$ actual J-hol. curves $\Phi(R, u_A, u_B, u_C) \in \mathcal{M}^{\$}_{2,4}(J, \dots)$ close to $u_A \#_R (u_B, u_C)$

some dimension

$\rightsquigarrow$ <u>gluing map</u> $\Phi : [R_0, \infty) \times \mathcal{M}_A \times (\mathcal{M}_B \times \mathcal{M}_C)/\mathbb{R} \longrightarrow \mathcal{M}^{\$}_{2,4}(J, A+B+C, \text{etc})$

s.t. $\lim_{R \to \infty} \Phi(R, u_A, u_B, u_C) \longrightarrow$ the building w/ comps u_A, u_B, u_C,

& every seq. of curves converging to this building is eventually in im Φ.

orientations:

thm 1: All mod. spaces w/ asymp. markers in $\hat{W}$ & $\mathbb{R}\times M_\pm$ can be oriented coherently, i.e. s.t. all gluing maps are orientation preserving.

($\Rightarrow$ possible to prove $\partial^2 = 0$ with integer coeffs!)

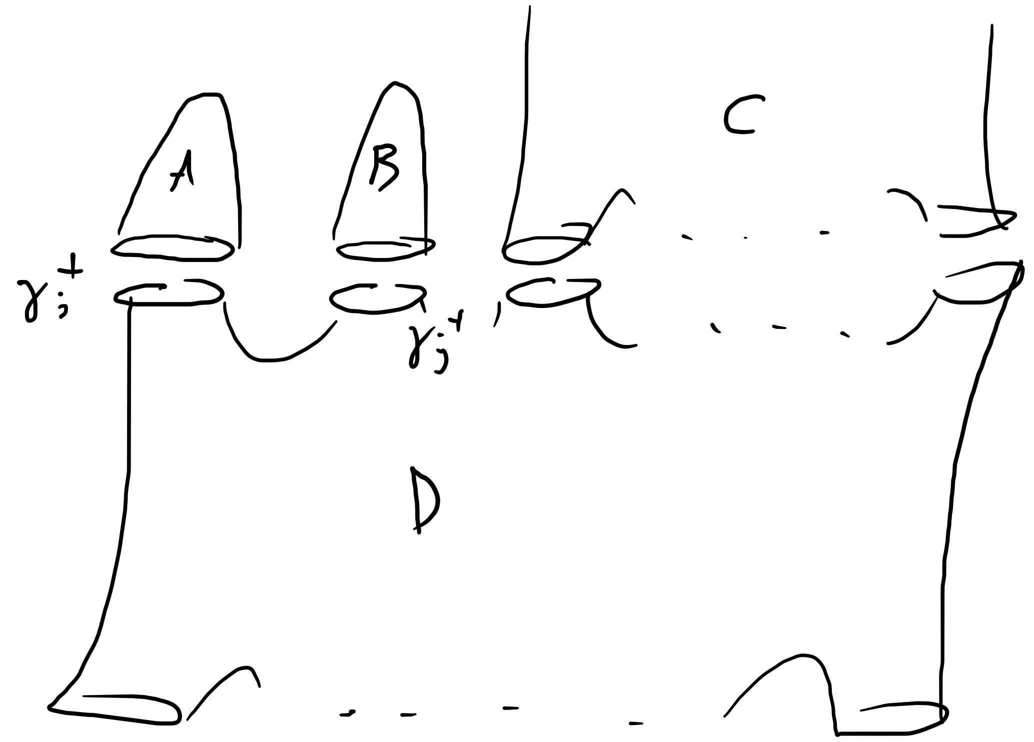
prop 1: Spse $\hat{\gamma}^+ = (\gamma_1^+, \dots, \gamma_{k_+}^+)$, $\check{\gamma}^+ := \hat{\gamma}^+$ w/ 2 orbits γ_i^+, γ_j^+ interchanged.

Then the map $\mathcal{M}^{\$}_{g,m}(J, A, \hat{\gamma}^+, \gamma^-) \longrightarrow \mathcal{M}^{\$}_{g,m}(J, A, \check{\gamma}^+, \gamma^-)$

def'd by interchanging order of 2 pts. reverses orientation iff

$n - 3 + \mu_{CZ}(\gamma_i^+)$ & $n-3+\mu_{CZ}(\gamma_j^+)$ are both odd.

pf: Imagine



$\Phi : [R, \infty) \times \mathcal{M}_A \times \mathcal{M}_B \times \mathcal{M}_C \times \mathcal{M}_D \longrightarrow \mathcal{M}^{\$}(\dots)$

orient. pres. $\Rightarrow$ the map that interchanges pts. is orient-reversing iff $\mathcal{M}_A \times \mathcal{M}_B \longrightarrow \mathcal{M}_B \times \mathcal{M}_A$, $(u_A, u_B) \longmapsto (u_B, u_A)$

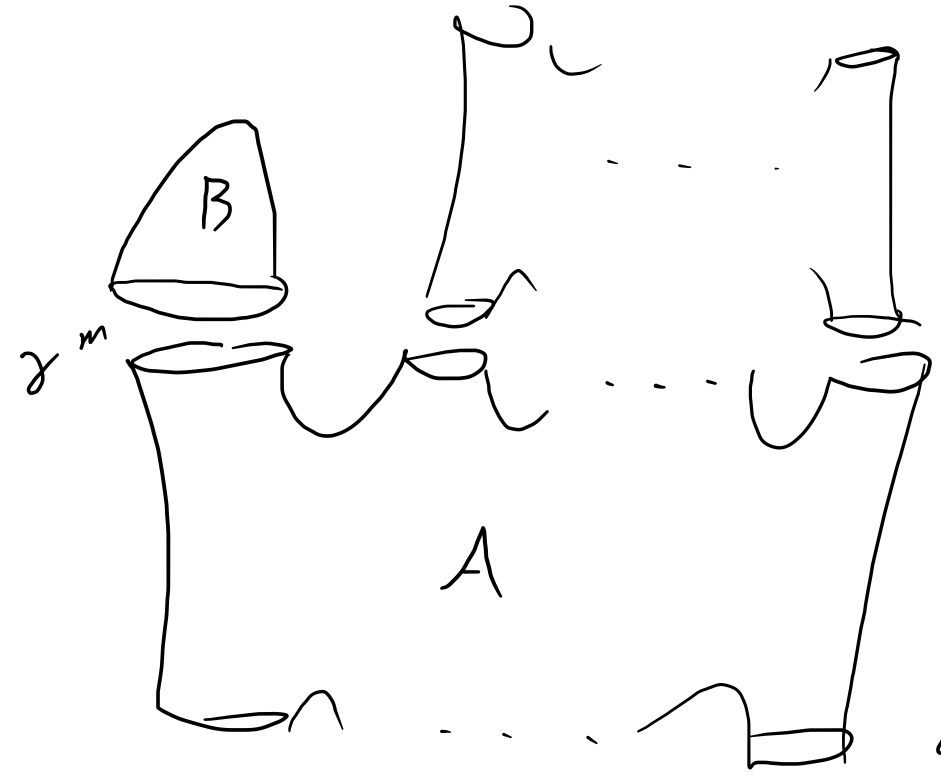
is orient. rev. $\Longleftrightarrow$ $\dim \mathcal{M}_A$ & $\dim \mathcal{M}_B$ both odd.

$$\dim \mathcal{M}_A = (n-3)\chi(\mathbb{C}) + 2c_1(A) - \mu_{CZ}(\gamma_i^+) = n - 3 + \mu_{CZ}(\gamma_i^+) \pmod 2. \quad \square$$

<u>prop 2</u>: Spse $\mathcal{M}^\#(J) \longrightarrow \mathcal{M}^\#(J)$ def'd by adjusting the asymp. marker at an m-fold covered orbit γ^m by mult. with $e^{2\pi i/m}$.
This reverses orientation iff m is even & $\mu_{CZ}(\gamma^m) - \mu_{CZ}(\gamma)$ is odd.

<u>defn</u>: γ^m satisfying both conds. is called a <u>bad</u> orbit. Others are <u>good</u>.

<u>pf of prop 2</u>:



Assume $u_B : \mathbb{C} \longrightarrow \mathbb{R}\times M_+$ is an m-fold cover:
$u_B(z) = v(z^m)$ for some
$v : (\mathbb{C}, i) \longrightarrow (\mathbb{R}\times M, J)$

Coherence $\Rightarrow$ change in orientation on rotating the marker at γ^m is the same for $\mathcal{M}_A$ & $\mathcal{M}_B$. This action on $\mathcal{M}_B$ is equivalent to $u_B \longmapsto u_B'(z) := u_B(e^{2\pi i/m} z)$. This will reverse orient. of $\mathcal{M}_B$ iff its linearization

$$\ker D_{u_B} \longrightarrow \ker D_{u_B} : \eta \longmapsto \eta'(z) := \eta(e^{2\pi i/m} z)$$

is orient. rev. This defns. a representation of $\mathbb{Z}_m$ on $\ker D_{u_B}$.

Repr. theory $\Rightarrow$ $\ker D_{u_B} \cong V_+ \oplus V_- \oplus V_{rot}$, where
$\mathbb{Z}_m$ acts on V_+ as Id, on V_- by $e^{2\pi i/m} \cong -\mathrm{Id}$ (only if m even),
$\mathbb{Z}_m$ acts on V_{rot} as a direct sum of rotations of 2-dim. spaces ($\Rightarrow$ orient.-pres.)
$\Rightarrow$ the action of $e^{2\pi i/m}$ reverses orient. iff $\dim V_-$ is odd.

Assume D_{u_B} & D_v both surj. $\quad \ker D_{u_B} \cong V_+ \oplus V_- \oplus V_{rot}$.

$$V_+ = \{ \eta \in \ker D_{u_B} \mid \eta(e^{2\pi i/m} z) = \eta(z) \ \forall z \} = \{ \eta(z) = \xi(z^m) \mid \xi \in \ker D_v \}$$

$$\cong \ker D_v .$$

$$\Rightarrow \dim V_- = \underbrace{\dim \ker D_{u_B}}_{\mathrm{ind}\, D_{u_B}} - \underbrace{\dim V_+}_{\mathrm{ind}\, D_v} - \underbrace{\dim V_{rot}}_{\text{even}} = \mathrm{ind}\, D_{u_B} - \mathrm{ind}\, D_v \pmod 2$$

$$= n\chi(\mathbb{C}) + \mu_{CZ}(\gamma^m) - [n\chi(\mathbb{C}) + \mu_{CZ}(\gamma)] = \mu_{CZ}(\gamma^m) - \mu_{CZ}(\gamma) \pmod 2.$$

$\square$