orienting $\mathcal{M}^{\mathfrak{S},\mathrm{reg}}(J)$ (closed case)

$\mathcal{M}_{g,m}(J,A) \ni [(\Sigma, j_0, \Theta, u_0)]$ has nbhd $\cong \bar\partial_J^{-1}(0) / \mathrm{Aut}(\Sigma, j_0, \Theta)$

w/ $\bar\partial_J : \mathcal{J} \times W^{k,p}(\Sigma, \hat W) \to \mathcal{E}^{k-1,p} : (j,u) \mapsto du + J(u) \circ du \circ j$

$\Rightarrow T_{u_0}\mathcal{M}(J) = \ker D\bar\partial_J(j,u) / \mathrm{aut}(\Sigma, j_0, \Theta)$. $\quad$ $\mathrm{aut}(\Sigma, j_0, \Theta)$ has a natural $\mathbb{C}$-str. $\rightsquigarrow$ orientation

$\Rightarrow$ orientation of $T_{u_0}\mathcal{M}(J)$ is equiv. to an orientation of $\ker D\bar\partial_J(j,u)$

$D\bar\partial_J(j_0,u_0) : \underbrace{T_{j_0}\mathcal{J}}_{\text{cpx vec. space}} \oplus W^{k,p}(u_0^*T\hat W) \to W^{k-1,p}(\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u_0^*T\hat W))$

$(y, \eta) \longmapsto \underbrace{J(u_0)\circ du_0 \circ y}_{\mathbb{C}\text{-linear}} + \underbrace{D_{u_0}}_{\text{CR-type op.}}\eta$

If D_{u_0} is $\mathbb{C}$-linear, $\ker D\bar\partial_J(j_0,u_0)$ is cpx V.S $\Rightarrow$ has natural orientation.

idea: D_{u_0} is homotopic through Fredholm ops. to its $\mathbb{C}$-linear part

$D^{\mathbb{C}}_{u_0}\eta := \frac{1}{2}\left(D_{u_0}\eta - J D_{u_0}(J\eta)\right)$, $\quad$ let $D^s_{u_0} := s D_{u_0} + (1-s) D^{\mathbb{C}}_{u_0}$, $\quad 0 \le s \le 1$.

$\rightsquigarrow L^s(y,\eta) = J(u_0)\circ du_0 \circ y + D^s_{u_0}\eta$.

fantasy: If L^s surj $\forall s \in [0,1]$, their kernels vary continuously,

$\Rightarrow$ can define an orientation of every $\ker D\bar\partial_J(j,u)$ by retracting the ops. to $\mathbb{C}$-linear parts & using natural orientation of $\ker L^0$.

problem: Cannot usually guarantee L^s surj. $\forall s \in [0,1]$.

determinant line bndl: Fix X, Y Banach spaces. For $T \in \mathrm{Fred}_{\mathbb{R}}(X,Y)$,

$\det(T) := \Lambda^{max} \ker T \otimes (\Lambda^{max} \operatorname{coker} T)^*$. Convention: $\Lambda^{max}\{0\} = \mathbb{R}$

$\Rightarrow$ if T is an iso, $\det(T) = \mathbb{R}$.

rk: $\mathcal{U} \overset{open}{\subseteq} \mathrm{Fred}_{\mathbb{R}}(X,Y)$ surj ops.

$\underset{T}{\overset{\Psi}{}} \rightsquigarrow \det(T) = \Lambda^{max} \ker T \Rightarrow$ these form a real rank 1 vec. bndl over $\mathcal{U}$ (since $\bigcup_{T\in\mathcal{U}} \ker T$ is a V.B.).

thm: $\exists$ a real topological line bndl $\det(X,Y) \xrightarrow{\pi} \mathrm{Fred}_{\mathbb{R}}(X,Y)$ s.t. $\pi^{-1}(T) = \det(T)$ & this bndl structure is compatible w/ the bndl str. of $\bigcup_{T\in\mathcal{U}} \ker T$ for $\mathcal{U} := \{T \text{ surj.}\}$.

observe: An orientation of $\det(T)$ is equiv. to an orient. $\ker T \oplus \operatorname{coker} T$; so $\ker T$ if T is surj.

pf for X, Y fin.-dim: claim: $\forall\ T \in \mathcal{L}_{\mathbb{R}}(X,Y)$, $\exists$ natural iso

$$\det(T) \xrightarrow{\Phi_T} \det(X \xrightarrow{0} Y) = \Lambda^{max} X \otimes (\Lambda^{max} Y)^*$$

$\Lambda^{max} \ker T \otimes \Lambda^{max}(\operatorname{coker} T)^*$

$0 \neq k = k_1 \wedge \dots \wedge k_p$, $0 \neq c^*$ where

($p = \dim \ker T$) $\quad c = c_1 \wedge \dots \wedge c_q$ ($c_i \in \operatorname{coker} T$)

$c^* \in (\Lambda^{max} \operatorname{coker} T)^*$ s.t. $c^*(c) = 1$.

$$\Phi_T(k \otimes c^*) = (k \wedge v_1 \wedge \dots \wedge v_m) \otimes (c \wedge \underbrace{Tv_1 \wedge \dots \wedge Tv_m}_{\text{basis of } \operatorname{im} T})^*$$

basis of $\operatorname{im} T \Rightarrow c \wedge Tv_1 \wedge \dots \wedge Tv_m$ is indep. of choice of an identification of $\operatorname{coker} T$ w/ a complement of $\operatorname{im} T \subseteq Y$.

for $v_1, \dots, v_m \in X$ any basis of a subspace complementary to $\ker T$.

This is indep. of choice of complements $V = \mathrm{Span}\{v_1, \dots, v_m\} \subseteq X$, (to $\ker T$)

$C = \mathrm{Span}\{c_1, \dots, c_q\} \subseteq Y$. (to $\operatorname{im} T$)

Fix V, C. claim: indep. of $v_1, \dots, v_q \in V$.

Write $v = v_1 \wedge \dots \wedge v_q$, $v' = v_1' \wedge \dots \wedge v_q' \neq 0 \in \Lambda^{max} V$, then $v' = \lambda v$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. $\Rightarrow$

$$(k \wedge v') \otimes (c \wedge Tv')^* = (k \wedge \lambda v) \otimes (c \wedge \lambda T v)^*$$
$$= \lambda (k \wedge v) \otimes (\lambda(c \wedge Tv))^*$$
$$= \lambda \tfrac{1}{\lambda} (k \wedge v) \otimes (c \wedge Tv)^* = \Phi_T(k \otimes c^*).$$

(by defn: $c^*(c) = 1$, $\frac{1}{\lambda} c^*(\lambda c) = 1 \Rightarrow (\lambda c)^* = \frac{1}{\lambda} c^*$.)

$\blacksquare$

pf in general: Given $T_0 \in \mathrm{Fred}_{\mathbb{R}}(X,Y)$, write $X = \left\{ \begin{matrix} V & \xrightarrow[\cong]{T_0} & W = \mathrm{im}\, T_0 \\ \oplus & & \oplus \\ K = \ker T_0 & & C \cong \mathrm{coker}\, T_0 \end{matrix} \right\} = Y$

Write $T \in \mathrm{Fred}_{\mathbb{R}}(X,Y)$ as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ & assume

T close enough T_0 s.t. $A : V \longrightarrow W$ is invtbl.

Write $F(T) := \begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} \in \mathcal{L}_{\mathbb{R}}(\underbrace{V \oplus K}_{X})$, $G(T) := \begin{pmatrix} 1 & 0 \\ -CA^{-1} & 1 \end{pmatrix} \in \mathcal{L}_{\mathbb{R}}(\underbrace{W \oplus C}_{Y})$

$\Phi(T) := D - CA^{-1}B \in \mathcal{L}_{\mathbb{R}}(K, C)$. $F(T), G(T)$ are invtbl,

$T' := G(T)\, T\, F(T) = \begin{pmatrix} A & 0 \\ 0 & \Phi(T) \end{pmatrix} : \underbrace{V \oplus K}_{X} \longrightarrow \underbrace{W \oplus C}_{Y}$

i.e. $T' = A \oplus \Phi(T)$ where A = iso., $\Phi(T) : K \longrightarrow C$.

$\Rightarrow$ natural iso. $\ker T' = \ker \Phi(T) \subseteq \ker T_0$, $\mathrm{im}\, T' = W \oplus \mathrm{im}\, \Phi(T)$

$\Rightarrow \mathrm{coker}\, T' = \mathrm{coker}\, \Phi(T)$.

EX: For any $T_i \in \mathrm{Fred}_{\mathbb{R}}(X_i, Y_i)$ with $i = 1, 2$, $\exists$ canonical iso.

$$\det(T_1 \oplus T_2) = \det(T_1) \otimes \det(T_2).$$

For any T near T_0, $\exists$ natural isos.

$$\det(T) \xleftarrow[\cong]{F(T)_* \otimes G(T)^*} \det(T') = \det(A \oplus \Phi(T)) = \det(A) \otimes \det(\Phi(T))$$

$\det(A) \otimes \det(\Phi(T)) = \det(\Phi(T))$ since A is an iso. ($\det(A) = \mathbb{R}$)

$$\det(T_0) = \Lambda^{max} \ker T_0 \otimes (\Lambda^{max} \mathrm{coker}\, T_0)^* = \det(K \xrightarrow{0} C) \xleftarrow[\cong]{\Psi_{\Phi(T)}} \det(\Phi(T))$$

$\rightsquigarrow$ local triv. for $\det(X,Y)|_{\mathrm{nbhd}(T_0)}$.

$\boxtimes$

prop: If X & Y are $\mathbb{C}$-vec. spaces, $\det(X,Y)\big|_{\mathrm{Fred}_{\mathbb{C}}(X,Y)}$ has a canonical orientation which is the natural orientation of $\mathbb{R}$ for all isos.

Pf: Call $\underset{\neq 0}{k} \in \Lambda^{max} \ker T$ complex if has form $k = k_1 \wedge i k_1 \wedge k_2 \wedge i k_2 \wedge \dots \wedge k_p \wedge i k_p$

for $k_1, \dots, k_p$ a cpx basis of $\ker T$.

Orient $\det(X,Y)\big|_{\mathrm{Fred}_{\mathbb{C}}(X,Y)}$ s.t. $\forall$ cpx elements $k \in \Lambda^{max} \ker T$, $c \in \Lambda^{max} \mathrm{coker}\, T$,

$k \otimes c^* \in \det(T)$ is positive.

For $T: X \to Y$ an iso., spse T is in domain of a local triv. of $\det(X,Y)$ near $T_0 : X \to Y$, T_0 is $\mathbb{C}$-linear but not an iso.

to show: if V, W are cpx vec. spaces & $A: V \xrightarrow{\cong} W$ $\mathbb{C}$-linear, then $\Psi_A : \mathbb{R} \longrightarrow \det(V \xrightarrow{\cong} W)$ is orient.-pres. (w.r.t. canonical orient. of $\mathbb{R}$ & cpx orient. of $\det A$).

$$\det(V \xrightarrow{\cong} W) = \Lambda^{max} V \otimes (\Lambda^{max} W)^*.$$

$\Psi_A(1) = v \wedge (Av)^*$ for any $v \neq 0 \in \Lambda^{max} V$.

Choose v to be cpx, then Av is also cpx $\Rightarrow$ $\Psi_A(1)$ represents the pos. orientation of $\det(V \xrightarrow{\cong} W)$. $\boxtimes$

cor: On a closed Riem surf., $\forall$ CR-type ops D, $\det(D)$ has a canonical orientation def'd by deformation to its $\mathbb{C}$-linear part.

$\rightsquigarrow$ canonical orientation of $\mathcal{M}^{\text{reg}}(J)$ for closed curves

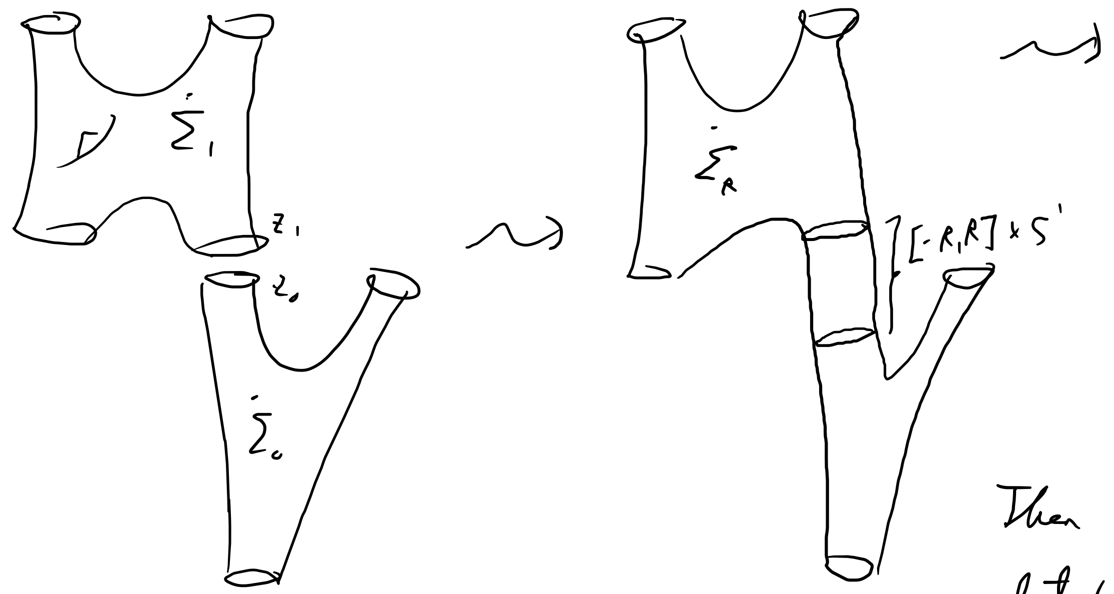
Bourgeois - Mohnke algorithm: Given an asymp. Herm. vec. bndl $(E, J) \to (\dot{\Sigma}, j)$ & asymp. ops. $\{A_z\}_{z \in \Gamma}$ (assume all nondeg.)

$CR(E, \{A_z\}) := \{$ CR-type ops on E asymp. to A_z at each $z \in \Gamma\}$

= affine space of Fredholm ops.

trouble: Maybe no op. in $CR(E, \{A_z\})$ is $\mathbb{C}$-linear.

lemma: Spse $E^i \to \dot{\Sigma}^i = \Sigma^i \setminus \Gamma_i$ $(i = 0,1)$ have pts. $z_0 \in \Gamma_0^+$, $z_1 \in \Gamma_1^-$ s.t. some unitary bndl iso. $E^1_{z_1} \to E^0_{z_0}$ identifies A_{z_1} w/ A_{z_0}.

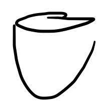


$\rightsquigarrow$ bndl $E^R \to \dot{\Sigma}^R$ for $R > 0$ def'd by gluing E^0 & E^1 along this iso. Can define a glued CR-op. D^R on E^R up to htpy.

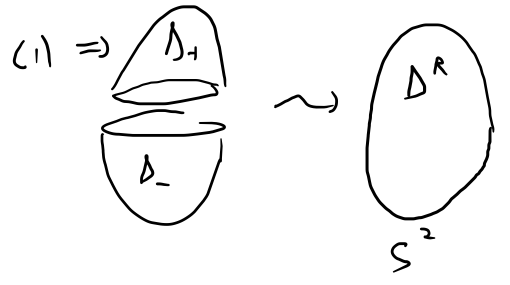
Then for $R >> 0$, $\exists$ natural iso.

$\det(D_0) \otimes \det(D_1) \to \det(D^R)$. $\boxed{}$

procedure: (1) Choose orientations arbitrarily for ops. on trivial bndles over  $= (S^2 \setminus \{\infty\}, i)$ w/ every possible asymp. op.

(2) Define orientation for ops. on trivial bndls over  s.t.

(1) $\Rightarrow$



$\det(D_-) \otimes \det(D_-) \cong \det(D^R)$ & $\det(D^R)$ carries the canonical "cpx" orientation

orientation pres.

(3) On any $\dot{\Sigma}$, define orientation s.t. capping

respects orientation & produces the cpx orientation over a closed surface.

thm: Orientations of $\mathcal{M}^*(J)$ def'd in this way are always coherent. $\boxed{}$