- symplectic linear algebra
- Darboux's thm + Moser trick + Gray's thm
- cont mfds

linear algebra: Spse $V = 2n$-dim. vec. space, $\omega \in \Lambda^2 V^*$ nondeg:

$\forall\ v \neq 0 \in V, \quad \omega(v, \cdot) \neq 0 \in V^*$.

lemma 1: $\exists$ a basis $e_1, f_1, \ldots, e_n, f_n \in V$ s.t. $\omega(e_i, f_i) = 1 \ \forall i$,

$\omega(e_i, f_j) = 0 \ \forall i \neq j$

$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \ \forall i, j$.

cor: ω nondeg. $\iff \omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$.

lemma 2: $\forall$ subspaces $W \subseteq V$,

$W^{\perp\omega} := \{ X \in V \mid \omega(X, Y) = 0 \ \forall Y \in W \}$. Then $W^{\perp\omega} \subseteq V$ is a subspace of complementary dim.: $\dim W + \dim W^{\perp\omega} = 2n$.

defn: W is a symplectic subspace if $W \cap W^{\perp\omega} = \{0\}$ ($\iff \omega|_W$ also nondeg.)

isotropic $\quad W \subseteq W^{\perp\omega} \quad (\iff \omega|_W = 0)(\Rightarrow \dim W \leq n)$

coisotropic $\quad W \supseteq W^{\perp\omega} \quad (\Rightarrow \dim W \geq n)$

Lagrangian if $W = W^{\perp\omega}$ $(\iff \omega|_W = 0$ & $\dim W = n)$.

$J_\tau(V,\omega) = \{J : V \to V \text{ linear} \mid J^2 = -I \text{ \& } \omega(X, JX) > 0 \ \forall X \neq 0 \in V\}$

$J(V,\omega) := \{J \in J_\tau(V,\omega) \mid \omega(JX, JY) = \omega(X,Y)\}$.

$\overset{\in}{J}$, let $\langle X, Y\rangle := \omega(X, JY) \Rightarrow \langle X, X\rangle > 0 \ \forall X \neq 0$,

$\langle Y, X\rangle = \omega(Y, JX) = \omega(JY, -X) = \omega(X, JY) = \langle X, Y\rangle \Rightarrow \langle\ ,\ \rangle$ is an inner product.

fundamental lemma: $J(V,\omega)$ is nonempty & contractible.

Pf: Let $\mathcal{M}(V) := \{\text{inner products on } V\}$, a nonempty convex set.

$\exists$ map $J(V,\omega) \to \mathcal{M}(V) : J \mapsto g_J := \omega(\cdot, J\cdot)$.

claim: This map admits a contin. left-inverse $\Phi : \mathcal{M}(V) \to J(V,\omega)$,

i.e. $\Phi(g_J) = J$.

idea: $g_J(X,Y) = \omega(X, JY) \Leftarrow \omega(X,Y) = g_J(JX, Y)$.

$\qquad\qquad \overset{\shortparallel}{-\omega(JY, X)} = \omega(Y, JX) = -\omega(JX, Y)$

$\forall g \in \mathcal{M}(V)$, $\exists!$ antisym. (wrt g) nonsingular map $A : V \to V$

s.t. $\omega = g(A\cdot, \cdot)$. Polar decomposition: $A = \underbrace{A\sqrt{A^*A}^{-1}}_{\text{orthogonal}} \underbrace{\sqrt{A^*A}}_{\text{symm. pos-def.}}$

check: $\Phi(g) := A\sqrt{A^*A}^{-1}$ is a $\mathbb{C}$-str.

compat. w/ ω, it is J if $g_J = g$.

$\square$

Darboux's thm: $\omega \in \Omega^2(M^{2n})$ closed & nondeg. $\Rightarrow \forall\, x \in M$, $\exists$ a nbhd $\mathcal{U} \subseteq M$ of x w/ coords. $(q,p) : \mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^n$ s.t. on $\mathcal{U}$,

$$\omega = \sum_{j=1}^n dp_j \wedge dq_j =: \omega_{std}.$$

pf (via Moser deformation trick): Lemma 1 $\Rightarrow$ can assume WLOG $M = \mathbb{R}^{2n}$, $x = 0$, $\omega = \omega_{std}$ at 0.

For $\tau \in [0,1]$, let $\omega_\tau := \tau\omega + (1-\tau)\omega_{std}$, so $\omega_0 = \omega_{std}$, $\omega_1 = \omega$,

$\omega_\tau = \omega_{std}$ at 0 $\forall\, \tau$ $\Rightarrow$ ω_τ is symplectic $\forall\, \tau$ on some nbhd of 0.

idea: Find a family of diffeos. $\{\varphi_\tau\}_{\tau \in [0,1]}$ btwn nbhds of 0, fixing 0, s.t. $\varphi_\tau^* \omega_\tau = \omega_0$ $\forall\, \tau$ also $\varphi_0 = \mathrm{Id}$. If exists, then $\varphi_1^* \omega = \omega_{std}$, done!

If φ_τ exists, it is the flow of a τ-dep. vec. fld X_τ def'd near 0.

Then $\omega_0 = \varphi_\tau^* \omega_\tau$ $\forall\, \tau$ $\Rightarrow 0 = \partial_\tau(\varphi_\tau^* \omega_\tau) = \varphi_\tau^*(\partial_\tau \omega_\tau) + \varphi_\tau^*(\mathcal{L}_{X_\tau} \omega_\tau)$

$= \varphi_\tau^*(\partial_\tau \omega_\tau + \mathcal{L}_{X_\tau} \omega_\tau) \Rightarrow \omega - \omega_{std} + d\iota_{X_\tau} \omega_\tau = 0$

Poincaré lemma $\Rightarrow$ $\omega = d\lambda$, $\omega_{std} = d\lambda_{std}$ for 1-forms s.t. $\lambda = \lambda_{std}$ at 0.

Now $0 = d(\lambda - \lambda_{std} + \iota_{X_\tau} \omega_\tau)$. Would suffice to find X_τ s.t.

$\lambda - \lambda_{std} + \omega_\tau(X_\tau, \cdot) = 0$, i.e. $\omega_\tau(X_\tau, \cdot) = \lambda_{std} - \lambda$.

ω_τ nondeg. $\Rightarrow \exists!\, X_\tau$ satisfying this, & $X_\tau = 0$ at 0 $\forall\, \tau$.

$\Rightarrow$ flow φ_τ is def'd on some nbhd of 0 $\forall\, \tau \in [0,1]$. $\boxtimes$

Contact mflds: M^{2n-1} oriented, $\alpha \in \Omega^1(M)$ s.t. $\alpha \wedge (d\alpha)^{n-1} > 0$, $\xi := \ker \alpha$.

Lemma 3: $\alpha \wedge (d\alpha)^{n-1} \neq 0 \iff (d\alpha)^{n-1}|_{\xi} \neq 0 \iff d\alpha|_{\xi}$ is nondeg.

$\Rightarrow (\xi, d\alpha)$ is a "symplectic vector bndl" over M.

$d\alpha$ can't be nondeg on $M \Rightarrow \exists!$ line bndl $\ell \subseteq TM$, $\ell = \ker d\alpha$

& $\ell \pitchfork \xi$. $\Rightarrow \exists!$ vec. fld $R_\alpha \in \mathfrak{X}(M)$ s.t. $\begin{cases} d\alpha(R_\alpha, \cdot) \equiv 0 \\ \alpha(R_\alpha) \equiv 1 \end{cases}$ Reeb vector fld.

R_α deps. on α, not just ξ.

equivalent statement of Weinstein conj: $\forall$ closed M w/ ctct form α, R_α has a periodic orbit.

Gray's stability thm: M closed, $\{\xi_\tau\}_{\tau \in [0,1]}$ fam. of ctct strs.

Then $\xi_\tau = (\varphi_\tau)_* \xi_0$ for some fam. of diffeos $\varphi_\tau : M \to M$, $\varphi_0 = \mathrm{Id}$.

Pf: $\xi_\tau = \ker \alpha_\tau$. If φ_τ exists, then $\varphi_\tau^* \alpha_\tau = f_\tau \alpha_0$ for some fam. of fns $f_\tau : M \to (0,\infty)$. Write $\dot f_\tau := \partial_\tau f_\tau : M \to \mathbb{R}$, $\dot\alpha_\tau := \partial_\tau \alpha_\tau \in \Omega^1(M)$.

If φ_τ = flow of X_τ, $\partial_\tau(\varphi_\tau^* \alpha_\tau) = \varphi_\tau^* \dot\alpha_\tau + \varphi_\tau^* \mathcal{L}_{X_\tau} \alpha_\tau$

$$= \varphi_\tau^*(\dot\alpha_\tau + \mathcal{L}_{X_\tau}\alpha_\tau) = \dot f_\tau \alpha_0 = \frac{\dot f_\tau}{f_\tau} f_\tau \alpha_0 = \frac{\dot f_\tau}{f_\tau} \varphi_\tau^* \alpha_\tau = \varphi_\tau^*\Big(\underbrace{\Big(\frac{\dot f_\tau}{f_\tau} \circ \varphi_\tau^{-1}\Big)}_{g_\tau : M \to \mathbb{R}} \alpha_\tau\Big)$$

$$\iff \dot\alpha_\tau + \mathcal{L}_{X_\tau}\alpha_\tau = g_\tau \alpha_\tau$$

$$= \dot\alpha_\tau + d\iota_{X_\tau}\alpha_\tau + \iota_{X_\tau} d\alpha_\tau$$

optimism: Try to find X_τ with values in ξ_τ. $\Rightarrow \alpha_\tau(X_\tau) \equiv 0$.

Then the needed relation becomes $\dot\alpha_\tau + d\alpha_\tau(X_\tau, \cdot) = g_\tau \alpha_\tau$.

Restriction to ξ_τ: $\dot\alpha_\tau|_{\xi_\tau} = -d\alpha_\tau(X_\tau, \cdot)|_{\xi_\tau}$ since $\alpha_\tau|_{\xi_\tau} = 0$

$d\alpha_\tau|_{\xi_\tau}$ nondeg. $\Rightarrow$ this uniquely determines X_τ.

Evaluate on R_{α_τ}: $\dot\alpha_\tau(R_{\alpha_\tau}) = g_\tau$ $\Rightarrow$ determines g_τ.

X_τ determines flow φ_τ, $\frac{\dot f_\tau}{f_\tau} = \partial_\tau(\ln f_\tau) = g_\tau \circ \varphi_\tau \Rightarrow f_\tau$ determined (up to scaling).

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