(1) Sobolev spaces via Fourier
(2) distributions
(3) "pf" of estimates for $\bar\partial$

Fourier transform

$f : \mathbb{R}^n \longrightarrow \mathbb{C}$, $\quad \mathcal{F}f =: \hat{f}$, $\quad f = \mathcal{F}^* \hat{f}$

$\check{g} := \mathcal{F}^* g$

$$\hat{f}(p) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i p\cdot x} \, d^n x$$

$$\partial_j f(x) = \int_{\mathbb{R}^n} \hat{f}(p) \partial_j e^{2\pi i p \cdot x} \, \underbrace{d^n p}$$

Lebesgue measure for fns of $p \in \mathbb{R}^n$

$\mathcal{F}, \mathcal{F}^* : L^1 \longrightarrow C^0$ bdd

$\mathcal{S}(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid x^\alpha \partial^\beta f$ is a bdd fn on $\mathbb{R}^n$ $\forall$ multi-indices $\alpha, \beta \}$

Then $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ & $\mathcal{F}^* : \mathcal{S} \longrightarrow \mathcal{S}$.

(1) Plancherel's thm: $\forall f, g \in \mathcal{S}$, $\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{L^2}$

$\Rightarrow$ $\mathcal{F}$ & $\mathcal{F}^*$ have unique extensions to contin. isometries $L^2 \longrightarrow L^2$.

(2) $\mathcal{F}(L^1) \subseteq C^0$, $\mathcal{F}^*$ same

(3) $\forall$ $f \in \mathcal{S}$ & multi-ind. α, $\widehat{\partial^\alpha f}(p) = (2\pi i p)^\alpha \hat{f}(p)$

(4) For $f, g \in L^1$ & $f * g(x) = \int_{\mathbb{R}^n} f(x) g(x-y) \, d^n y$, $\quad \widehat{f * g} = \hat{f} \hat{g}$.

Formally if $f \in L^2$ & $\partial^\alpha f \in L^2$ $\forall$ $|\alpha| \le k$ $\Longleftrightarrow$

$\forall$ $|\alpha| \le k$, the fn. $g_\alpha(p) := p^\alpha \hat{f}(p)$ defines a fn $g_\alpha \in L^2$.

$$\Longleftrightarrow \quad \|f\|^2_{H^k} := \int_{\mathbb{R}^n} (1+|p|^2)^k \, |\hat{f}(p)|^2 \, d^n p < \infty$$

$\rightsquigarrow$ alternate defn. of Sobolev spaces $W^{k,2}(\mathbb{R}^n) =: H^k(\mathbb{R}^n)$,

$f \in L^2(\mathbb{R}^n)$ s.t. $\|f\|_{H^k} < \infty$. (<u>note</u>: makes sense $\forall$ $k \in \mathbb{R}$)

<u>Sobolev emb. thm</u>: If $2k > n$, then $H^k \hookrightarrow C^0$.

<u>pf</u>: Suff. to prove $\|\hat{f}\|_{L^1} \le c\|f\|_{H^k}$ $\forall$ $f \in H^k$.

$$|\hat{f}| = \underbrace{(1+|p|^2)^{k/2}|\hat{f}|}_{\in L^2} \cdot \frac{1}{(1+|p|^2)^{k/2}} \quad \overset{(\text{Hölder})}{\Longrightarrow}$$

$$\|\hat{f}\|_{L^1} \lesssim \|f\|_{H^k} \cdot \left(\int_{\mathbb{R}^n} \frac{d^n p}{(1+|p|^2)^k} \right)^{1/2} \longleftarrow \text{integral conv. iff } 2k > n.$$

$\boxtimes$

<u>rk</u>: Can also defn $H^k(\mathbb{T}^n)$ $\forall$ $k \in \mathbb{R}$ via Fourier series.

$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

e.g. Conley-Zehnder ~1984: ∇-flow in $H^{1/2}(S^1)$ (see Hofer-Zehnder book)

distributions: $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $\mathcal{D}(U) := C_0^\infty(U)$ w/ topology s.t. ("test fns")

$\varphi_k \to \varphi$ if $\exists$ $K \subseteq U$ cpt s.t. $\mathrm{supp}(\varphi_k) \subseteq K$ & $\varphi_k \xrightarrow{C^\infty} \varphi$.

$\mathcal{D}'(U) := \{\text{contin linear fnals } l : \mathcal{D}(U) \to \mathbb{R}\}$ "distributions"

ex 1: $f \in L^1_{loc}(U) \Rightarrow l_f \in \mathcal{D}'(U)$ def'd by $l_f(\varphi) := \int_U f\varphi$.

ex 2: If $0 \in U$, $\delta(\varphi) := \varphi(0)$. "Dirac δ-fn." "$\int_U \delta \cdot \varphi = \varphi(0)$."

defn: For $l \in \mathcal{D}'$, write $l(\varphi) =: (l, \varphi)$, can defn. derivative $\partial_j l \in \mathcal{D}'$ by $(\partial_j l, \varphi) = -(l, \partial_j \varphi)$.

EX: $f \in \mathcal{D}'(\mathbb{R})$ def'd via the fn $f(x) = |x| \Rightarrow f' \in \mathcal{D}'(\mathbb{R})$ is given by $f'(x) = \mathrm{sgn}(x)$. $\quad f'' = 2\delta$.

thm: The convolution naturally extends to $\mathcal{D}' \times \mathcal{D} \to \mathcal{D}'$ s.t.

(i) $\partial_j(l * f) = \partial_j l * f = l * \partial_j f$, $\quad$ (ii) $\delta * f = f$

"$\int_{\mathbb{R}^n} \delta(x-y) f(y)\, d^n y = f(x)$."

<u>prop</u>: Let $K(z) := \frac{1}{2\pi z}$, so $K \in L^1_{loc}(\mathbb{C}, \mathbb{C})$. Then $\bar\partial K = \delta$.

<u>Pf</u>: To show, $\forall\ \varphi \in C_0^\infty(\mathbb{C})$, $\varphi(0) = (\bar\partial K, \varphi) = -(K, \bar\partial\varphi) = -\int_{\mathbb{C}} K(z)\,\bar\partial\varphi(z)\,dm(z)$.

Recall $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial \bar z}$ are $\mathbb{C}$-lin. combins. of ∂_s, ∂_t def'd via chain rule:

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\bar\partial, \quad \frac{\partial}{\partial z} = \frac{1}{2}\partial, \quad \text{Lebesgue measure } dm(z) = \frac{dz \wedge d\bar z}{-2i} = ds \wedge dt. \quad \begin{pmatrix} z = s+it \\ \bar z = s-it \end{pmatrix}$$

Now
$$-\int_{\mathbb{C}} \frac{1}{2\pi z}\bar\partial\varphi(z)\,dm(z) = -\int_{\mathbb{C}} \frac{1}{\pi z}\frac{\partial\varphi}{\partial\bar z}\frac{dz\wedge d\bar z}{-2i} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{z}\frac{\partial\varphi}{\partial\bar z}\,dz\wedge d\bar z$$

$$= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{z}\frac{\partial\varphi}{\partial\bar z}\,d\bar z\wedge dz = -\frac{1}{2\pi i}\int_{\mathbb{C}} d\left(\frac{\varphi(z)}{z}dz\right) = -\frac{1}{2\pi i}\lim_{\varepsilon\searrow 0}\left(\int_{D_{\frac{1}{\varepsilon}}\setminus D_\varepsilon} d\left(\frac{\varphi(z)}{z}dz\right)\right)$$

$$= -\frac{1}{2\pi i}\lim_{\varepsilon\searrow 0}\left(\underbrace{\int_{\partial D_{\frac{1}{\varepsilon}}}\frac{\varphi(z)}{z}dz}_{=0 \text{ since } \operatorname{supp}(\varphi) \text{ cpct}} - \int_{\partial D_\varepsilon}\frac{\varphi(z)}{z}dz\right) = \varphi(0).$$

$\boxtimes$

cor: $\forall\ f \in C_0^\infty(\mathbb{C})$, $\bar\partial(K * f) = \bar\partial K * f = \delta * f = f$

i.e. K a "fundamental sol." to the eqn. $\bar\partial u = f$.

thm: For $1 < p < \infty$, $\exists\ c > 0$ s.t. $\forall\ f \in C_0^\infty(\mathbb{D})$,

(1) $\|K * f\|_{L^p(\mathbb{D})} \le c\|f\|_{L^p}$, — minor adaptation of Young's ineq. (Fubini + Hölder)

(2) $\|\partial(K * f)\|_{L^p(\mathbb{D})} \le c\|f\|_{L^p}$ — REALLY HARD (except if $p = 2$)

(3) $\|\underbrace{\bar\partial(K * f)}_{= f}\|_{L^p(\mathbb{D})} \le c\|f\|_{L^p}$. trivial

$\partial(K * f) = K * \partial f$ useless for estimates in terms of $\|f\|_{L^p}$.
$= \partial K * f$

EX: $\partial K \in \mathcal{D}'(\mathbb{C})$ can be identified with the (not locally integrable) fn. $-\frac{1}{\pi z^2}$ understood as a distribution by $(\partial K, \varphi) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{C} \setminus \mathbb{D}_\varepsilon} -\frac{1}{\pi z^2} \varphi(z)\, dm(z)$

Then $(\partial K * f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_{|\zeta - z| \ge \varepsilon} \frac{f(\zeta)}{(z-\zeta)^2}\, dm(\zeta)$ singular integral operator

Calderón–Zygmund ineq. $\Rightarrow$ (2) in general.

<u>case $p=2$</u>: $\bar\partial$ is a 1st-order linear op. $\Rightarrow$

F.T. identifies it w/ mult. by a linear polynomial on $\mathbb{C}$

$$\widehat{\bar\partial u}(\zeta) = P(\zeta)\,\hat u(\zeta) \quad \text{for a polynomial } P(\zeta) := 2\pi i\,\zeta$$

$P(\zeta) \neq 0 \quad \forall\, \zeta \neq 0 \quad$ ($\Leftarrow$ $\bar\partial$ is an <u>elliptic</u> operator)

$\Rightarrow \; |P(\zeta)| \gtrsim c|\zeta| \quad \forall\, \zeta$ near 0, $c>0$. (const.)

Now $\|\partial(K*f)\|_{L^2(\mathbb{D})} \le \|\partial(K*f)\|_{L^2(\mathbb{C})} = \|\widehat{\partial(K*f)}\|_{L^2} = \|2\pi i\,\bar\zeta\,\widehat{K*f}\|_{L^2}$

$\le c\,\|P(\zeta)\widehat{K*f}\|_{L^2} = c\,\|\widehat{\bar\partial(K*f)}\|_{L^2} = c\,\|\bar\partial(K*f)\|_{L^2} = c\,\|f\|_{L^2}$. $\boxtimes$

<u>cor</u> (by density of $C_0^\infty \subseteq L^p$): $f \mapsto K*f$ extends to bdd lin. op.

$L^p \to W^{1,p}$ s.t. $\bar\partial(K*f) = f \quad \forall\, f \in L^p$.

<u>cor</u>: $\forall\, u \in W_0^{1,p}(\mathbb{D})$, $\quad \|u\|_{W^{1,p}} \le c\,\|\bar\partial u\|_{L^p}$.

Pf: <u>EX</u>: $\forall\, f \in C_0^\infty$, $|K*f|$ decays as $\frac{1}{|z|}$ near ∞.

Spse $u \in C_0^\infty$, $\bar\partial u = f \in C_0^\infty$, then $\bar\partial(u - K*f) \equiv 0 \Rightarrow u - K*f$ is

a hol. fn. on $\mathbb{C}$ decaying at ∞, $\Rightarrow u \equiv K*f$. $\quad \|u\|_{W^{1,p}} = \|K*f\|_{W^{1,p}} \le c\,\|\underset{\bar\partial u}{\underset{\shortparallel}{f}}\|_{L^p}$.

C_0^∞ dense in $W_0^{1,p}$. $\boxtimes$