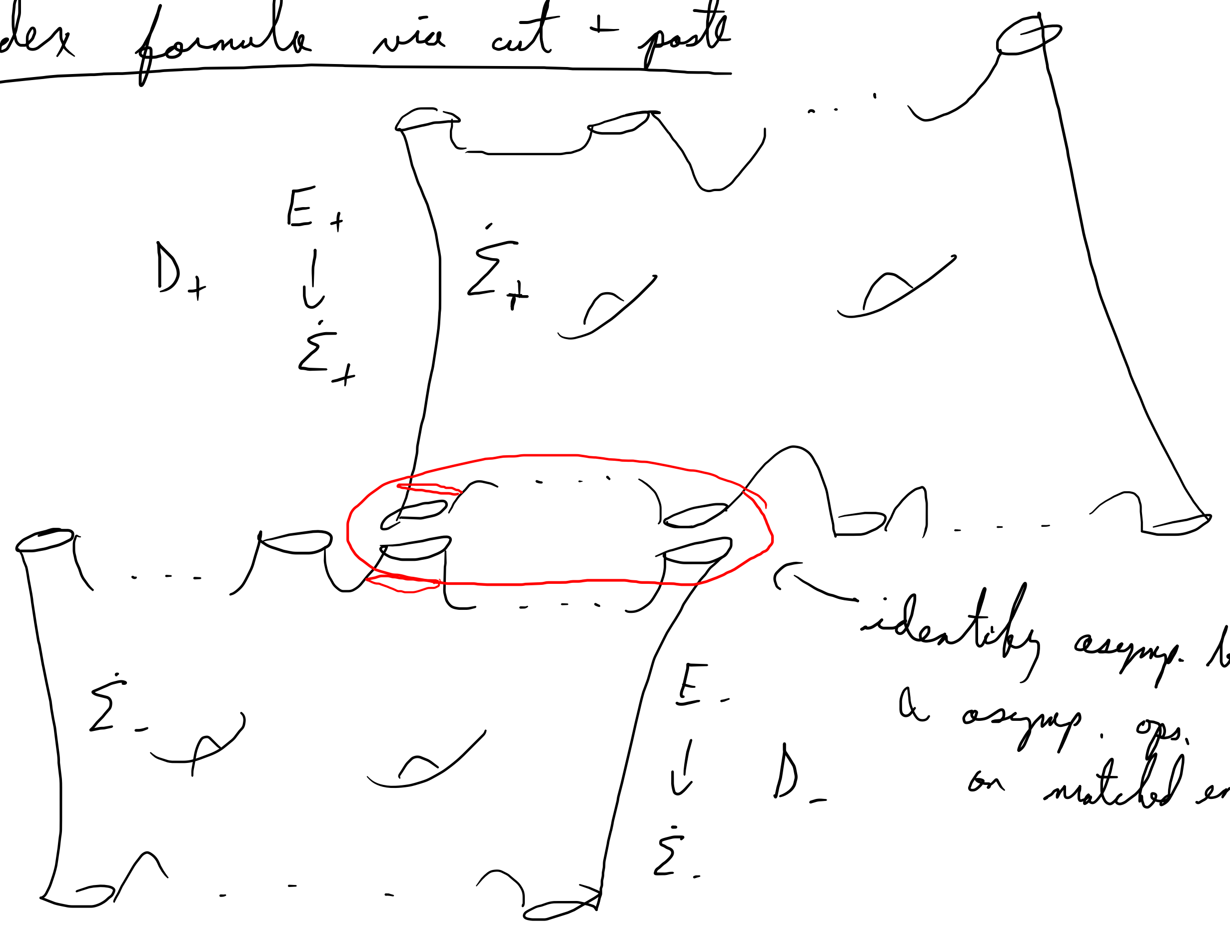
index formula via cut + paste



glue together along matching ends $\rightsquigarrow$ new surface $\dot\Sigma$

identify asymp. bundles & asymp. ops. on matched ends

$\rightsquigarrow$ CR-op. D on bundle $E \to \dot\Sigma$ obtained by gluing bundles E_+ & E_- along matching asymp. bundle (unique up to deformation on a cpct subset)

thm (M. Schwarz):

$$\mathrm{ind}(D) = \mathrm{ind}(D_+) + \mathrm{ind}(D_-).$$

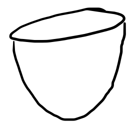
(pf by linear gluing)

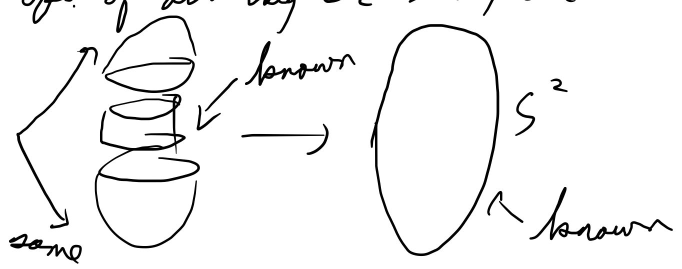
cor: Suff. to prove index formula for the following special cases:

(1) trivial line bundle over S^2

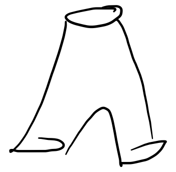
(2) " " " over $\mathbb{R}\times S^1$ w/ asymp. ops. of arbitrary CZ-index at each end.

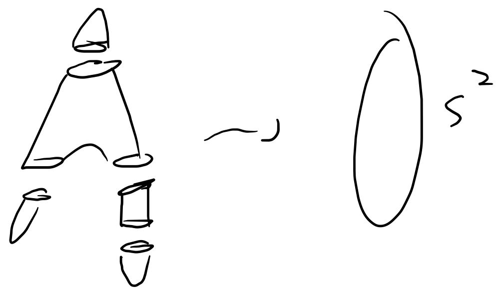
pf: (i) Can do bundles over $\mathbb{C}$:



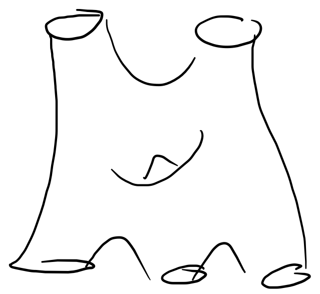


(ii) Can do bundles over





(iii)



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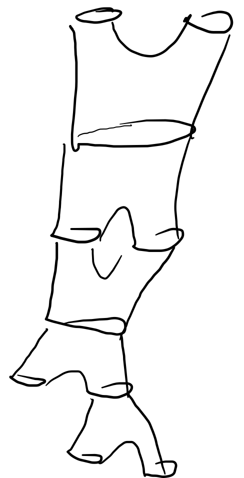

$\rightsquigarrow$ everything.

<u>Case S^2</u>: $E \to S^2$ trivial line bundl, $D = \bar\partial$

$\Rightarrow \ker D = \{\text{const. fns } S^2 \to \mathbb{C}\}$, $\dim_{\mathbb{R}} \ker D = 2$.

<u>claim</u>: D is surj. ($\Leftarrow D^*$ is inj).

- D^* as anti-CR op. on $F = \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, \overset{\overset{\text{trivial}}{\shortparallel}}{E}) \cong TS^2 \otimes \text{trivial}$

$\Rightarrow c_1(F) = c_1(TS^2) = \chi(S^2) = 2$.

- D^* is CR-op. on $\bar F \cong F^*$; $\quad F^* \otimes F \xrightarrow{\cong}$ trivial

$$\lambda \otimes v \longmapsto \lambda(v)$$

$\Rightarrow c_1(\bar F) = -2$.

If $\xi \neq 0 \in \ker D^*$. $\quad \# \xi^{-1}(0) = c_1(\bar F) = -2$

but similarity principle $\Rightarrow$ all zeroes of ξ are isolated & positive

contra!

$\Rightarrow \dim \operatorname{coker} D = 0 \Rightarrow \operatorname{ind} D = 2 - 0 = 2 = \chi(S^2) + 2c_1(E)$. $\blacksquare$

Case $\overset{(s,t)}{\mathbb{R}\times S^1}$: consider $D = \bar\partial + S(t) + f'(s) : H^1(\mathbb{R}\times S^1, \mathbb{C}) \to L^2(\mathbb{R}\times S^1, \mathbb{C})$

for some fn. $f:\mathbb{R}\to\mathbb{R}$ s.t. $f(s) = \begin{cases} bs & \text{for } s \gg 1 \\ as & \text{for } s \ll -1 \end{cases}$.

Let $A = -i\partial_t - S(t)$, then D is asymp. at $+\infty$ to

$A-b$, at $-\infty$ to $A-a$.

$\Rightarrow$ index of D should be $\mu_{CZ}(A-b) - \mu_{CZ}(A-a) = \mu^{spec}(A-a, A-b)$

$= \#$ e-vals of A in (a,b) if $b>a$,

or $-$(same) if $a>b$.

notice: for $u:\mathbb{R}\times S^1\to\mathbb{C}$, $D(e^{-f(s)}u)$

$= \partial_s(e^{-f}u) + (i\partial_t + S + f')(e^{-f}u) = -f'(s)e^{-f}u + e^{-f}(\bar\partial u + Su + f'(s)u)$

$= e^{-f}(\bar\partial u + Su)$.

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\cdot e^{-f}} & \Gamma(E) \\ \downarrow{\scriptstyle \bar\partial+S} & & \downarrow{\scriptstyle D} \\ \Gamma(F) & \xrightarrow{\cdot e^{-f}} & \Gamma(F) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow[\hat=]{\cdot e^{-f}} & H^1 \\ \downarrow{\scriptstyle \bar\partial+S} & & \downarrow{\scriptstyle D} \\ Y & \xrightarrow[\hat=]{\cdot e^{-f}} & L^2 \end{array}$$

$X := \{u:\mathbb{R}\times S^1\to\mathbb{C} \mid e^{-f(s)}u\in H^1\}$ $\quad \|u\|_X := \|e^{-f}u\|_{H^1}$

$Y := \{u \mid e^{-f(s)}u \in L^2\}$ $\quad \|u\|_Y := \|e^{-f}u\|_{L^2}$

$u\in X \iff u\in H^1_{loc}(\mathbb{R}\times S^1)$, $e^{-bs}u\in H^1(Z_+)$, $e^{-as}u\in H^1(Z_-)$

similarly for Y with L^2-norms.

Need to compute kernel of $\bar{\partial} + S = \partial_s - A : X \to Y$.

What do sols. to $\partial_s u - Au = 0$ look like locally?

Consider $u_s := u(s, \cdot) : S^1 \to \mathbb{C}$.

<u>lemma</u>: $L^2(S^1, \mathbb{C})$ is spanned by O-N eigenfns. of A.

Pf: Pick $\lambda \in \mathbb{R} \setminus \sigma(A)$, then

$$\begin{tikzcd} & & H^1 \arrow[dr, "\text{cpt}"] & \\ (\lambda - A)^{-1} : & L^2 \arrow[ur] \arrow[rr, "\text{cpt}"'] & & L^2 \end{tikzcd}$$

is a cpt self-adjoint op. w/ same eigenfns as A. ∎

<u>note</u>: For any $\lambda \in \sigma(A)$ & eigenfn. $\eta_\lambda : S^1 \to \mathbb{C}$,

$u_\lambda(s,t) := e^{\lambda s} \eta_\lambda(t)$ is a sol. to $\partial_s u - Au = 0$.

$\Rightarrow$ all sols. on domains of the form $(s_-, s_+) \times S^1$ are linear combinations of these!

Q: For which $\lambda \in \sigma(A)$ do the sols. $u_\lambda(s,t) = e^{\lambda s}\eta_\lambda(t)$ belong to X?

A: All are in $H^1_{loc}(\mathbb{R} \times S^1)$.

$$e^{-bs} u_\lambda = e^{-(b-\lambda)s}\eta_\lambda \in H^1(Z_+) \quad \text{iff} \quad \lambda < b$$

$$e^{-as} u_\lambda = e^{(\lambda - a)s}\eta_\lambda \in H^1(Z_-) \quad \text{iff} \quad \lambda > a$$

$\Rightarrow$ $\exists$ 1 dimension of sols. for every e-val. (counted w/ multiplicity) $\lambda \in \sigma(A)$ with $a < \lambda < b$,

$\Rightarrow$ If $a < b$, then $\dim \ker D = \mu^{spec}(A-a, A-b)$,

if $b < a$, D is injective.

Do some argument for D^*

$\Rightarrow$ if $a < b$, D^* is injective,

if $b < a$, $\dim \ker D^* = \#$ e-vals. of $-A$ in $(-a, -b)$

$$= \mu^{spec}(-A+a, -A+b) = -\mu^{spec}(A-a, A-b).$$

$\Rightarrow$ In both cases, $\operatorname{ind}(D) = \mu^{spec}(A-a, A-b)$.

rk: Can choose A, $a, b \in \mathbb{R} \setminus \sigma(A)$ to make $\mu_{CZ}(A-a)$ & $\mu_{CZ}(A-b)$ whatever we want.