## $\infty$ - dim. calculus:

refs: S. Lang

$U \overset{\text{open}}{\subseteq} X$ Banach, $Y$ Banach. $f: U \to Y$ diff-able at $x \in U$ if

$\exists$ $df(x) \in \underbrace{\mathcal{L}(X,Y)}_{\text{bdd lin maps}}$ s.t. $f(x+h) = f(x) + df(x)h + \|h\|_X \cdot R(h)$

where $\lim_{h \to 0} R(h) = 0$.

$\rightsquigarrow$ $df: U \to \mathcal{L}(X,Y)$ = another Banach space $\Rightarrow$ can inductively defn. $C^k(U,Y)$ for $k \geq 0$.

chain rule: $f, g \in C^k \Rightarrow f \circ g \in C^k$, $d(f \circ g)(x) = df(g(x))\, dg(x)$

product rule: $Y, Z, V$ Banach spaces, $\mu: Y \times Z \to V$ contin. bilinear map

$f: U \to Y$, $g: U \to Z$ diff-able at $x \in U$ $\Rightarrow$

$d(\mu \circ (f,g))(x)h = \mu(df(x)h, g(x)) + \mu(f(x), dg(x)h)$.

partial derivs: $U \overset{\text{open}}{\subseteq} X \times Y$, $f: U \to Z$, $d_1 f(x,y) \in \mathcal{L}(X,Z)$

$d_2 f(x,y) \in \mathcal{L}(Y,Z)$

derivs. of $f$ w.r.t. 1 variable w/ other held const.

If both exist & are contin. on a nbhd of $(x,y)$, then $f$ is diff-able at $(x,y)$, $df(x,y)(v,w) = d_1 f(x,y)v + d_2 f(x,y)w$.

linear fns: $f(x) = Ax$ for some $A \in \mathcal{L}(X,Y)$, then $df(x) = A$ $\forall x$.

$\Rightarrow$ $d(df)(x) = 0$ $\quad (\Rightarrow f \in C^\infty)$

mean value thm: $f \in C^1(U,Y)$, $x \in U$, $h \in X$ s.t. $x + th \in U$ $\forall t \in [0,1]$,

$$f(x+h) = f(x) + \int_0^1 \frac{d}{dt} f(x+th)\, dt \overset{f(x)+}{=} \int_0^1 \underbrace{df(x+th)}_{\in \mathcal{L}(X,Y)} h\, dt$$

$$= f(x) + \underbrace{\left( \int_0^1 df(x+th)\, dt \right)}_{\in \mathcal{L}(X,Y)} h.$$

inverse fn. thm: $f \in C^k(\mathcal{U}, Y)$ $(k \geq 1)$ s.t. $f(x) = y$, $df(x): X \to Y$ is a Banach space iso. Then $f$ maps a nbhd of $x$ bijectively to a nbhd of $y$, & $f^{-1}|_{\text{nbhd}} \in C^k$.

Pf: Contraction mapping princ. (AKA Banach fixed pt. thm).

implicit fn. thm: $f \in C^k(\mathcal{U}, Y)$ $(k \geq 1)$ s.t. $f(x_0) = y_0$, $df(x_0): X \to Y$ surj. & has a bdd right-inverse ($\Leftrightarrow \exists$ splitting of closed subspaces $X = \ker df(x_0) \oplus V$)

Then $\exists$ nbhd $\mathcal{O} \subseteq K := \ker df(x_0)$ of $0$ & $\varphi \in C^k(\mathcal{O}, \mathcal{U})$ embedding s.t. $\varphi(\mathcal{O})$ is a nbhd of $x_0$ in $f^{-1}(y_0)$.

Pf: $F: \mathcal{U} \to Y \times K : x \longmapsto (f(x), \pi_K(x - x_0))$ for some bdd lin. proj. map $\pi_K : X \to K$.

$dF(x_0) = (df(x_0), \pi_K) : X \to Y \times K$ iso. $\Rightarrow F$ locally invtbl,

$\varphi = F(0, \cdot)$. ☒

defn: $C^k$-smooth Banach mfd: locally homeo. to open subset of a Banach space, transition maps of class $C^k$ (+ Hausdorff etc: usually metrizable + separable)

submfd: $X$ a Banach space, $M$ a Banach mfd $\subseteq X$ s.t. inclusion $M \xhookrightarrow{i} X$ smooth, homeo. onto its image, $\forall x \in M$, $Ti : T_xM \to X$ is inj. & image has closed complement.

$(\Leftrightarrow)$ near each $x \in M$, $\exists$ diffeo nbhd$(x) \cong$ open subset of $Y \times Z$ s.t. $X \cong Y \times \{0\}$.

application: $(M^{2n}, J)$ an almost cpx mfd, $p \in M$.

thm: $\forall$ $X \in T_pM$ suff. small, $\exists$ a $J$-hol. disk $u : (\mathbb{D}, i) \to (M, J)$, where $\mathbb{D} \ni s+it$,

s.t. $u(0) = p$, $\partial_s u(0) = X$.

Coord. choices $\Rightarrow$ suff. to prove: $\forall$ $X \in \mathbb{C}^n$, $\forall$ $J$ on $\mathbb{C}^n$

suff. $C^k$-close to $i$ for some $k \in \mathbb{N}$, $\exists$ $J$-hol. $u : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$

w/ $u(0) = 0$ & $\partial_s u(0) = X$.

pf idea: Fix $k \geq 2$, $p > 2$, so Sobolev $\Rightarrow$ $W^{k,p}(\mathring{\mathbb{D}}) \hookrightarrow C^1(\mathbb{D})$.

$m \in \mathbb{N}$, $\mathcal{O}^m :=$ a nbhd of $i$ in $C^m(\mathbb{D}^{2n}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$.

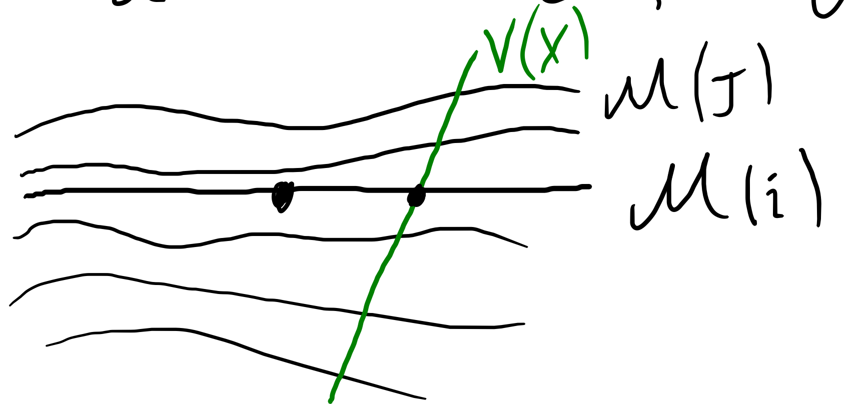
For $J \in \mathcal{O}^m$, $\mathcal{M}(J) := \{ u \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n) \mid \overline{u(\mathring{\mathbb{D}})} \subseteq \mathbb{D}^{2n},\ \underbrace{\bar{\partial}_J u := \partial_s u + J(u)\partial_t u}_{=0} \}$.

$\mathcal{M} := \{ (J, u) \in \mathcal{O}^m \times W^{k,p} \mid u \in \mathcal{M}(J) \}$.

$V(X) := \{ u \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n) \mid \partial_s u(0) = X \}$ = affine subspace of $W^{k,p}(\mathring{\mathbb{D}})$.

claim: If $m$ suff. large, then $\mathcal{M}$ is a $C^1$-submfd of $\mathcal{O}^m \times W^{k,p}$

& $\forall$ $J \in \mathcal{O}^m$, $\mathcal{M}(J) \subseteq W^{k,p}$ is a $C^1$-submfd $\pitchfork$ $V(X)$.



<u>main step</u>: $\mathcal{M}(J) = \bar\partial_J^{-1}(0)$ for $\bar\partial_J u := \partial_s u + (J \circ u)\,\partial_t u$.

Need to show $\bar\partial_J : W^{k,p} \to W^{k-1,p}$ is $C^1$ & $\forall\, u \in \bar\partial_J^{-1}(0)$,

$d\bar\partial_J(u) : W^{k,p} \to W^{k-1,p}$ has a bdd right inverse.

Case $J = i$: $\bar\partial_J = \bar\partial$ linear, $\Rightarrow d\bar\partial_i(u) = \bar\partial\ \forall\, u$, has a bdd right inverse.

$\Rightarrow$ also true $\forall\, J$ suff. close to $i$.

<u>Q</u>: How smooth is $\bar\partial_J : W^{k,p} \to W^{k-1,p}$ ?

(1) $u \mapsto \partial_s u : W^{k,p} \to W^{k-1,p}$ is contin. linear $\Rightarrow C^\infty$.

(2) $u \mapsto \partial_t u$ same.

(3) $(k-1)p > 2 \Rightarrow W^{k-1,p} \times W^{k-1,p} \to W^{k-1,p} : (J \circ u, \partial_t u) \mapsto (J \circ u)\,\partial_t u$

is a contin. bilinear map $\Rightarrow C^\infty$.

(4) How smooth is $W^{k,p} \to W^{k-1,p} : u \mapsto J \circ u$ ?

**Recall:** $C^k \times W^{k,p} \longrightarrow W^{k,p} : (f,u) \longmapsto f \circ u$ is contin. if $kp > n =$ dim. of domain

**Lemma:** For $r \in \mathbb{N}$, $kp > n$, $f \in C^{k+r}$, the map

$$\Phi_f : W^{k,p} \longrightarrow W^{k,p} : u \longmapsto f \circ u \quad \text{is of class } C^r \text{ \&}$$

$$d\Phi_f(u)h = (df \circ u)h.$$

(Note: $df \in C^k \implies df \circ u \in W^{k,p}$
$\implies (df \circ u)h \in W^{k,p}$ due to contin. product pairing $W^{k,p} \times W^{k,p} \longrightarrow W^{k,p}$.)

**Pf:** By induction, suff. to prove
$d\Phi_f(u) = df \circ u$.

$$\Phi_f(u+h)(z) = f \circ (u+h)(z) = f \circ u(z) + \int_0^1 \frac{d}{dt} f \circ (u+th)(z)\, dt$$

$$= \Phi_f(u)(z) + \left[\int_0^1 df \circ (u+th)(z)\, dt\right] h(z)$$

$$= \Phi_f(u)(z) + [df \circ u(z)]h(z) + [\theta_f \circ (u+h,u)(z)]h(z)$$

where we defn. $\theta_f(x,y) := \int_0^1 [df((1-t)y + tx) - df(y)]\, dt$.

$f \in C^{k+1} \implies \theta_f \in C^k \implies$

$$\Phi_f(u+h) = \Phi_f(u) + (df \circ u)h + \underbrace{\underbrace{[\overset{C^k}{\theta_f} \circ \underbrace{(u+h,u)}_{W^{k,p}}]}_{W^{k,p}}\, \overset{W^{k,p}}{h}}_{W^{k,p}}$$

$\theta_f(x,x) = 0 \;\forall x \implies$ by continuity of $W^{k,p} \times W^{k,p} \longrightarrow W^{k,p} : (u,v) \longmapsto \theta_f \circ (u,v)$,

$\lim_{h \to 0} \|\theta_f \circ (u+h,u)\|_{W^{k,p}} = 0$, $\|[\theta_f \circ (u+h,u)]h\|_{W^{k,p}} \leq c\|\theta_f \circ (u+h,u)\|_{W^{k,p}} \|h\|_{W^{k,p}}$

$\implies [\theta_f \circ (u+h,u)]h = o(\|h\|_{W^{k,p}})$. ∎