$\nexists$ Übung next week! (17.6.2020)

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rk: Thm. does not apply in general for cases with $\pi_\xi \circ du \not\equiv 0$ but $\mathcal{H}$ not contact.

ex: (W, Ω) closed symp. mfd, $M = \underset{t}{S^1} \times W$, $\mathcal{H} = (\Omega, dt)$

$J \in \mathcal{J}(\mathcal{H}) \iff \{J_t \in \mathcal{J}(W, \omega)\}_{t \in S^1}$. $\forall$ consts. $s \in \mathbb{R}$, $t \in S^1$,
if $v : (\Sigma, j) \longrightarrow (W, J_t)$ J_t-hol., then
$$u : (\Sigma, j) \longrightarrow (\mathbb{R} \times M, J)$$
$$z \longmapsto (s, t, v(z))$$
is J-hol.

sub-ex: $W = \Sigma_g$ s.t. $\mathrm{Id} : (\Sigma_g, J_t) \to (\Sigma_g, J_t)$ is hol. & cannot be perturbed away by perturbing $J_t \Rightarrow u$ also cannot.

$$\mathrm{ind}(u) = \underbrace{(n-3)}_{=-1} \chi(\Sigma) + 2c_1(\underbrace{u^* T(\mathbb{R} \times M)}_{= \text{ trivial} \oplus T\Sigma_g}) = \chi(\Sigma_g) = 2 - 2g < 0$$
if $g \geq 2$.

cor: $\forall$ perts. of J in $\mathcal{J}(\mathcal{H})$, $\exists$ embedded non-regular curves in $\mathcal{M}(J)$ with $\pi_\xi \circ du \not\equiv 0$.

Lemma (slightly wrong): Any path $\{A_s := -i\partial_t - S_s(t) : H^1(S^1,\mathbb{C}^n) \to L^2(S^1,\mathbb{C}^n)\}$ for asymp. ops. for $s \in [-1,1]$ admits a C^∞-small pert. of the form $\{A_s + B(s)\}_{s\in[-1,1]}$ for $B(s) \in \mathcal{L}^{sym}(L^2(S^1,\mathbb{C}^n))$ with $B(-1) = B(1) = 0$ s.t. $\forall\ s \in (-1,1)$, all e-vals of $A_s + B(s) : L^2 \supseteq H^1 \to L^2$ are simple.

"pf": Want to show: after perturbing A_s, the map

$$(-1,1) \times \mathbb{R} \longrightarrow \mathcal{L}^{sym}(H^1, L^2) : (s,\lambda) \longmapsto A_s - \lambda$$

never intersects the submfds $\underbrace{\{T \in \mathcal{L}^{sym}(H^1,L^2) \mid \dim\ker T = k\}}_{\text{codim } \frac{k(k+1)}{2} \text{ in } \mathcal{L}^{sym}(H^1,L^2), \text{ i.e. } \geq 3 \text{ if } k\geq 2.}$ for $k \geq 2$.

Suff. to show: the map is $\pitchfork$ to these submfds.

Let $\mathcal{B}^\varepsilon := \{B \in C^\infty([-1,1], \mathcal{L}^{sym}(L^2,L^2)) \mid \|B\|_{C_\varepsilon} < \infty,\ B(1) = B(-1) = 0\}$, is a Banach space.

not separable! contains an isometrically embedded copy of l^∞

For $k \in \mathbb{N}$, $\exists$ universal moduli space

$$\mathcal{M}_k(\mathcal{B}^\varepsilon) := \{(B,s,\lambda) \in \mathcal{B}^\varepsilon \times (-1,1) \times \mathbb{R} \mid (A_s + B(s) - \lambda) \text{ has kernel of dim. } k\}.$$

For $B \in \mathcal{B}^\varepsilon$, $\mathcal{M}_k(B) := \{(s,\lambda) \in (-1,1)\times\mathbb{R} \mid \dim(A_s + B(s) - \lambda) = k\}$

Near $(s_0, \lambda_0) \in \mathcal{M}_k(B)$, $\exists$ a nbhd $\mathcal{O} \subseteq (-1,1) \times \mathbb{R}$ of (s_0, λ_0) & a smooth map $\Phi : \mathcal{O} \longrightarrow \mathrm{End}^{\mathrm{sym}}(\ker(A_s + B(s) - \lambda))$ s.t. $\Phi^{-1}(0) = \mathcal{M}_k(B) \cap \mathcal{O}$.

Similarly near $(B_0, s_0, \lambda_0) \in \mathcal{M}_k(\mathcal{B}^\varepsilon)$, $\exists$ $\mathcal{O} \subseteq \mathcal{B}^\varepsilon \times (-1,1) \times \mathbb{R}$ & C^∞-map $\Phi : \mathcal{O} \longrightarrow \mathrm{End}^{\mathrm{sym}}(\ker(A_s + B(s) - \lambda))$ s.t. $\Phi^{-1}(0) = \mathcal{M}_k(\mathcal{B}^\varepsilon) \cap \mathcal{O}$.

<u>claim</u>: $d\Phi(B_0, s_0, \lambda_0)$ is surjective.

<u>pf</u>: $d_1 \Phi(B_0, s_0, \lambda_0) B' =: L_{B'} \in \mathrm{End}^{\mathrm{sym}}(\ker(A_s + B(s) - \lambda))$,

$L_{B'} \eta = \mathrm{proj}(B' \eta)$ for L^2-proj $L^2(S^1) \to \ker(A_s + B(s) - \lambda)$ . . . ∎

$\Rightarrow$ $\mathcal{M}_k(\mathcal{B}^\varepsilon)$ is a smooth Banach submfd of codim $\frac{k(k+1)}{2}$ in $\mathcal{B}^\varepsilon \times (-1,1) \times \mathbb{R}$.

Consider proj. $\mathcal{M}_k(\mathcal{B}^\varepsilon) \xrightarrow{\pi} \mathcal{B}^\varepsilon : (B, s, \lambda) \longmapsto B$.

<u>claim</u>: If $B \in \mathcal{B}^\varepsilon$ is a reg. val. of π, then $\mathcal{M}_k(B) \subseteq (-1,1) \times \mathbb{R}$ is a submfd of codim $\frac{k(k+1)}{2}$ ($\Rightarrow \emptyset$ if $k \geq 2$).

<u>Lemma</u>: X, Y, Z vector spaces, $D : X \to Z$, $A : Y \to Z$,

$L : X \oplus Y \to Z : (x, y) \longmapsto Dx + Ay$ surjective

$\Rightarrow$ for $\Pi : \ker L \to Y : (x, y) \longmapsto y$, $\exists$ natural isos.

$\ker \Pi \cong \ker D$, $\quad \mathrm{coker}\, \Pi \cong \mathrm{coker}\, D$.

In our situation D = derivative of $(-1,1) \times \mathbb{R} \overset{\text{open}}{\supseteq} \mathcal{O} \xrightarrow{\Phi} \mathrm{End}^{\mathrm{sym}}(\ker(A_s + B(s) - \lambda))$

$\Rightarrow$ D is Fredholm. fin.-dim

L = deriv. of $\mathcal{B}^\varepsilon \times (-1,1) \times \mathbb{R} \overset{\text{open}}{\supseteq} \mathcal{O} \xrightarrow{\Phi} \mathrm{End}^{\mathrm{sym}}(\cdots)$,

$\ker L = T_{(B_0, s_0, \lambda_0)} \mathcal{M}_k(\mathcal{B}^\varepsilon)$, $\quad \Pi$ = deriv. of proj. $\pi : \mathcal{M}_k(\mathcal{B}^\varepsilon) \to \mathcal{B}^\varepsilon$.

$\Rightarrow$ $d\pi(B_0, s_0, \lambda_0)$ is Fredholm & if regular, then Φ cuts $\mathcal{M}_k(B)$ out of $(-1,1) \times \mathbb{R}$ transversely $\Rightarrow$ codim $\mathcal{M}_k(B) = \frac{k(k+1)}{2}$.

Sard-Smale $\Rightarrow$ a comeager subset of $\mathcal{B}^\varepsilon$ consists of reg. vals. of π. ∎