Why $HF_*(V,\Omega) \cong H_*(V)$ (up to a degree shift)

Assume (V,Ω) a cpct symplectically aspherical symp. mfd

$\left(\int_{S^2} u^*\Omega = 0 \quad \forall\, u : S^2 \to V\right)$

e.g. Ω is exact

$H : V \times S^1 \to \mathbb{R}$, $\quad H_t := H(\cdot, t)$

$\{J_t \in \mathcal{J}(V,\Omega)\}_{t\in S^1} \rightsquigarrow$ Ham. vec. fld X_t, s.t. $dH_t = -\Omega(X_t, \cdot)$

Floer eqn: $(F)\quad \partial_s v + J_t(v)(\partial_t v - X_t(v)) = 0 \quad$ for $v : \mathbb{R}\times S^1 \to V$.

thm: Assume H & J are t-indep., $H : V \to \mathbb{R}$ is Morse w/ no crit. pts. on ∂V, & its ∇-flow w.r.t. $g := \Omega(\cdot, J\cdot)$ is Morse-Smale. Then if H is suff. C^2-small,

(1) Every $x \in \mathrm{Crit}(H)$ is a nondeg. 1-periodic orbit of $X_t =: X$ whose asymp. operator A_x satisfies $\boxed{\mu_{CZ}(A_x) = \mathrm{Morse}(x) - n.}$

(2) $\gamma : \mathbb{R} \to V$ w/ $\dot\gamma = -\nabla H(\gamma) \rightsquigarrow$ Fredholm regular sol. $v(s,t) := \gamma(s)$ to (F).

(3) All 1-per. orbits are as in (1) & all fin. energy sols. to (F) are as in (2).

cor: For these choices of data, the chain cpx of FH is identified w/ the chain cpx of Morse homology $\Rightarrow HF_*(V,\Omega) \cong H_*(V)$ (up to degree shift) pf: Recall: (F) is $\frac{d}{ds} v(s,\cdot) = -\nabla \mathcal{A}_H(v(s,\cdot))$

where for $\gamma: S^1 \to V$, $\nabla\mathcal{A}_H(\gamma) := J_t(\gamma)(\dot\gamma - X_t(\gamma)) \in \Gamma(\gamma^*TV)$,

so for $\gamma \in \mathrm{Crit}(\mathcal{A}_H)$, $\{\gamma_\rho \in C^\infty(S^1,V)\}_{\rho\in\mathbb{R}}$ s.t. $\gamma_0=\gamma$, $\partial_\rho \gamma_\rho|_{\rho=0} =: \eta \in \Gamma(\gamma^*TV)$,

$\Rightarrow A_\gamma \eta := \nabla_\rho(\nabla\mathcal{A}_H(\gamma_\rho))|_{\rho=0} = \boxed{J_t(\gamma)(\nabla_t\eta - \nabla_\eta X_t)}$

symmetric conn. on V — in ctct setting, minus sign goes here.

In our situation, $x\in \mathrm{Crit}(H)$, $\gamma(t)=x$, $dH = -\omega(X,\cdot) = g(\nabla H,\cdot)$

$= \omega(\nabla H, J\cdot) = -\omega(J\nabla H,\cdot) \Rightarrow X = J\nabla H$.

$\Gamma(\gamma^*TV) = C^\infty(S^1, T_xV)$, $A_\gamma\eta = J\partial_t\eta - \nabla_\eta(JX) = J\partial_t\eta + \underbrace{\nabla_\eta\nabla H}_{\text{Hessian of } H \text{ at } x.}$

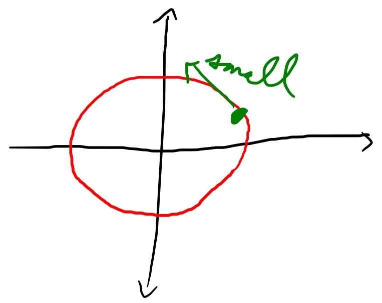
$A_\gamma = J(x)\partial_t + \nabla^2H(x)$.

$= i\partial_t + S$ in coords, for some $S\in \mathrm{End}^{sym}_{\mathbb{R}}(\mathbb{R}^{2n})$, invertible since crit. pt. is Morse.

$\eta\in\ker A_\gamma \Leftrightarrow i\dot\eta + S\eta = 0 \Leftrightarrow \dot\eta = iS\eta \Leftrightarrow \eta(t) = \exp(tiS)\eta(0)$ is periodic w/ period 1.

Note: $Y(\eta) := iS\eta$ is the Hamiltonian vec. fld on $\mathbb{R}^{2n}$ w.r.t. $K(\eta) = \frac{1}{2}\langle\eta, S\eta\rangle$.

fact (see Hofer-Zehnder): For any time-indep. Ham. system w/ a Hamiltonian that is C^2-small, all 1-per. orbits are constant.



consequence: Unless $\ker S \neq \{0\}$, $\nexists$ nontrivial 1-per. solution to $\dot\eta = iS\eta \Rightarrow A_\gamma$ is nondeg.

Choose a path $\{S_s \in \mathrm{End}^{sym}_{\mathbb{R}}(\mathbb{R}^{2n})\}_{s\in[0,1]}$ close to 0 s.t. $S_1 = S$, $S_0 = \begin{pmatrix}\epsilon I & \\ & -\epsilon I\end{pmatrix}$

By defn., for $A_s := i\partial_t + S_s$, $\mu_{CZ}(A_0) = 0$,

$\mu(A_\gamma)$ differs from that by $\mu^{spec}(A_0,A_1) = \mu^{spec}(S_0,S)$

since $\ker A_s \cong \ker S_s$ (constant e-fns).

$\mu^{spec}(S_0,S)$ is the difference btwn Morse index of $x\in\mathrm{Crit}(H)$ (= # neg. evals of S) & # neg. e-vals $S_0 = \epsilon\begin{pmatrix}I & \\ & -I\end{pmatrix}$, i.e. n.

(2) $u(s,t) = \gamma(s)$ solves (F).

Morse-Smale $\Rightarrow$ linearization of ∇-flow eqn is a surjective op.

Show: linearization of (F) is equivalent to that operator.

(3) To show: if H is replaced by cH for a const. $c > 0$ suff. small, then all sols. to (F) are t-indep., i.e. $v(s,t) = \gamma(s)$ for a ∇-flow line γ.

~~Pf~~: Spse $c_k \to 0$ pos. & $\exists$ non-S^1-inv. sols.

$v_k : \mathbb{R} \times S^1 \to V$ to $\partial_s v_k + J(v_k)(\partial_t v_k - c_k X(v_k)) = 0$.

WLOG can assume all have same const. asymp. orbits.

(For simplicity, also assume $\operatorname{ind}(v_k) = 1$).

Choose $N_k \in \mathbb{N}$ s.t. $N_k \to \infty$, $N_k c_k \to c > 0$, then let

$w_k : \mathbb{R} \times S^1 \to V$, $w_k(s,t) := v_k(N_k s, N_k t)$.

These satisfy $\partial_s w_k + J(w_k)(\partial_t w_k - N_k c_k X(w_k)) = 0$.

<u>claim</u>: $|dw_k|$ unif. bdd.

pf: If not, $\exists$ seq of rescaled maps $u_k : \mathbb{D}_{\varepsilon_k R_k} \to V$ satisfying a PDE that converges to the nonlinear CR-eqn., $E(u_k) < \infty$, C^1-bdd $\Rightarrow$ subseq. conv. to a nonconst J-hol. $u_\infty : \mathbb{C} \to V$.

$E(u_\infty) < \infty \Rightarrow u_\infty$ extends to a nonconst J-hol. sphere $u_\infty : S^2 \to V$.

Impossible since sympl. aspherical. $\boxtimes$

$w_k : \mathbb{R} \times S^1 \longrightarrow V$ are now C^1-bdd $\Longrightarrow$ a subseq. conv. to

$w_\infty : \mathbb{R} \times S^1 \longrightarrow V$ satisfying $\partial_s w_\infty + J\left(\partial_t w_\infty - c X(w_\infty)\right) = 0$.

WLOG $c > 0$ is arbitrarily small.

Since v_k is t-periodic & $N_k \longrightarrow \infty$, w_∞ is t-inv't.

($\Rightarrow$ w_k has period $\frac{1}{N_k}$ in t)

$\Rightarrow$ $w_\infty(s,t) = \gamma(s)$ for some ∇-flow line γ

$\overset{(2)}{\Longrightarrow}$ w_∞ is Fredholm regular: it lives in a smooth moduli space

w/ same dimension as the corresponding space of ∇-flow lines.

$\Longrightarrow$ Under any S^1-inv't perts. of the data, only S^1-inv't. sols.

(i.e. ∇-flow lines) can converge to w_∞. Contra. since we

assumed v_k <u>not</u> S^1-inv't. $\square$