Gromov - Witten invts (zero - point)

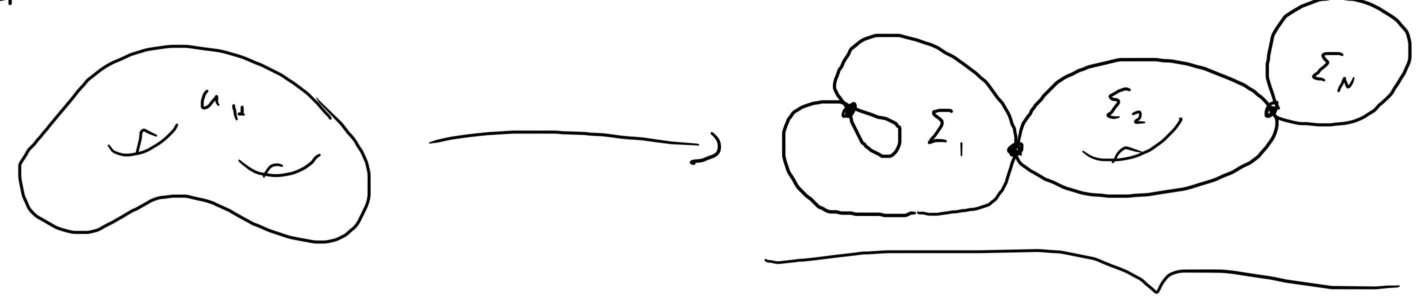
(W^{2n}, ω) closed $\qquad$ vir-dim $\mathcal{M}_{g,0}(J,A) = (n-3)(2-2g) + 2\underbrace{c_1(A)}_{\langle c_1(TW,J),\, A\rangle}$

$A \in H_2(W)$, $J \in \mathcal{J}(W,\omega)$ $\qquad = 0 \iff \boxed{c_1(A) = (n-3)(g-1).}$

assumption (science fiction): transversality is always satisfied

thm: $\mathcal{M}_{g,0}(J,A)$ is finite.

Pf: discrete since vir-dim $=0$; need to show cpt.



$S = \coprod_{i=1}^{N} \Sigma_i$

$u|_{\Sigma_i} := u_\infty^i$

$u_\infty = [(S, j, \Theta, \Delta, u)]$

For $i = 1, \dots, N$, let

$\hat{u}_i \in \mathcal{M}_{g_i, m_i}(J, A_i)$ denote u_∞^i w/ a marked pt. at every pt. in $\Sigma_i \cap \Delta$

$\Delta = \{\{z_1^+, z_1^-\}, \dots, \{z_p^+, z_p^-\}\}$ $\qquad \sum_{i=1}^{N} m_i = 2p = 2(\#\text{ nodes})$

$$\mathcal{M}_{g_1,m_1}(J,A_1) \times \dots \times \mathcal{M}_{g_N,m_N}(J,A_N) \xrightarrow{ev} \underbrace{W \times \dots \times W}_{2p}$$

Let $D := \{(w_1, \dots, w_{2p}) \in W^{\times 2p} \mid w_i = w_j$ whenever the ith & jth factors correspond to a node $\{z_k^+, z_k^-\} \in \Delta\}$

$D \subseteq W^{\times 2p}$ has codim. $= 2np$.

$(\hat{u}_1, \dots, \hat{u}_N) \in ev^{-1}(D)$. Assume $ev \pitchfork D$, so $ev^{-1}(D)$ is a mfd of

$$\dim = \sum_{i=1}^{N} \left[\text{vir-dim } \mathcal{M}_{g_i,m_i}(J,A_i)\right] - 2np$$

$\left(\underline{\text{note}}: \sum_{i=1}^{N}(2 - 2g_i - m_i) = 2 - 2g \qquad \sum_{i=1}^{N} A_i = A\right)$

$$= \sum_{i=1}^{N}\left[(n-3)(2-2g_i) + 2c_1(A_i) + 2m_i\right] - 2np = \sum_{i=1}^{N}\left[(n-3)(2-2g_i-m_i) + (n-3)m_i + 2c_1(A_i) + 2m_i\right] - 2np$$

$$= (n-3)(2-2g) + (n-3)2p + 2c_1(A) + 4p - 2np = \text{vir-dim } \mathcal{M}_{g,0}(J,A) + [2(n-3)+4-2n]p$$

$$= \text{vir-dim } \mathcal{M}_{g,0}(J,A) - 2(\#\text{ nodes}).$$

Since vir-dim $\mathcal{M}_{g,0}(J,A) = 0$, $ev^{-1}(D) = \emptyset$ unless # nodes $= 0$. $\Rightarrow$ compactness. $\square$

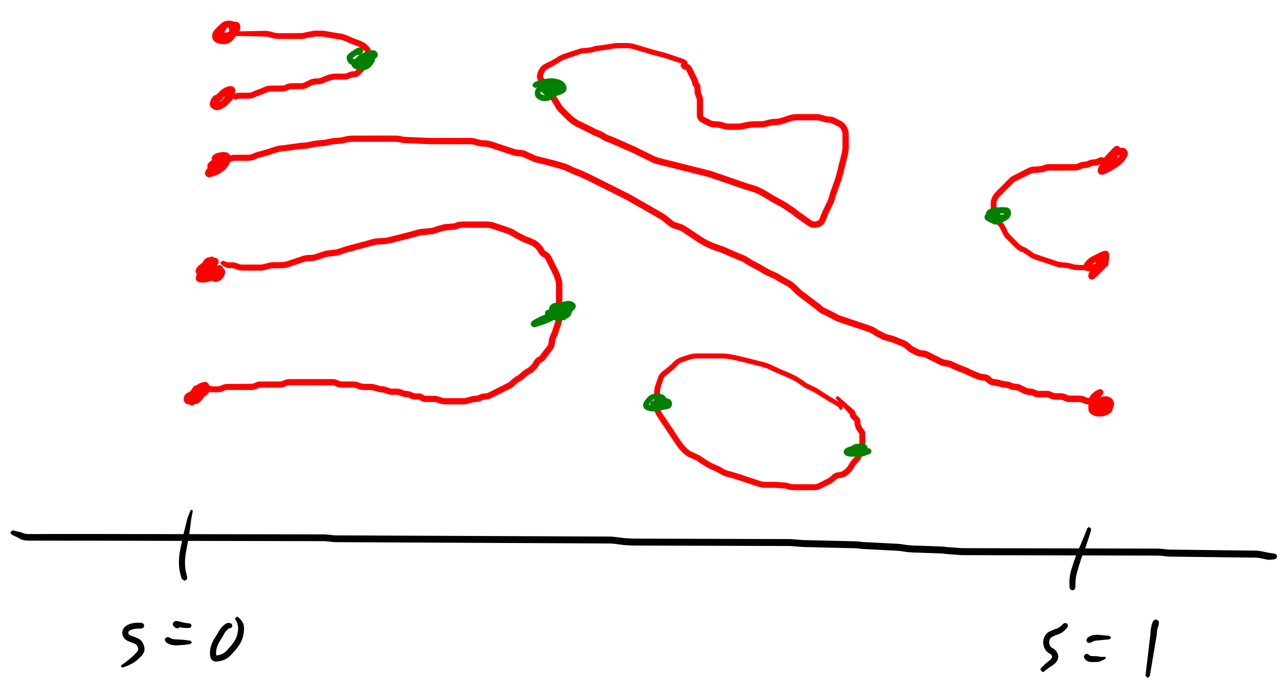
defn: $GW^{(W,\omega)}_{g,0,A} := \sum_{u \in \mathcal{M}_{g,0}(J,A)} \epsilon(u) \in \mathbb{Z}$ where $\epsilon(u) := \pm 1$ depending on the canonical orientation of $\mathcal{M}_{g,0}(J,A)$.

independence of J: Given $J_0, J_1 \in \mathcal{J}(W,\omega)$, choose generic fam. $\{J_s \in \mathcal{J}(W,\omega)\}_{s\in[0,1]}$

$\rightsquigarrow$ parametric moduli space $\mathcal{M}_{g,0}(\{J_s\},A) := \{(u,s) \mid s \in [0,1],\ u \in \mathcal{M}_{g,0}(J_s,A)\}$.

Assuming Λ + trivial aut. grps., $\mathcal{M}_{g,0}(\{J_s\},A)$ is a smooth cpct oriented 1-mfd w/ $\partial\mathcal{M}_{g,0}(\{J_s\},A) = -\mathcal{M}_{g,0}(J_0,A) \amalg \mathcal{M}_{g,0}(J_1,A)$

$\Rightarrow GW^{(W,\omega)}_{g,0,A}$ is the same for J_0 & J_1



thm: If $n = 2$ (i.e. $\dim W = 4$), $g = 0$ & $A \in H_2(W)$ is primitive, then for any generic $J \in \mathcal{J}(W,\omega)$, $GW^{(W,\omega)}_{g,0,A}$ is the actual number of J-hol. curves in $\mathcal{M}_{g,0}(J,A)$, i.e. $\epsilon(u) = +1 \ \forall u$.

"pf": Lemma 1: For generic J, non-immersed J-hol. curves (in $\dim W \geq 4$) live in a submfd of codim ≥ 2 in the moduli space

$\Rightarrow$ if vir-dim $\mathcal{M}_{g,0}(J,A)$, then all $u \in \mathcal{M}_{g,0}(J,A)$ are immersed for generic J.

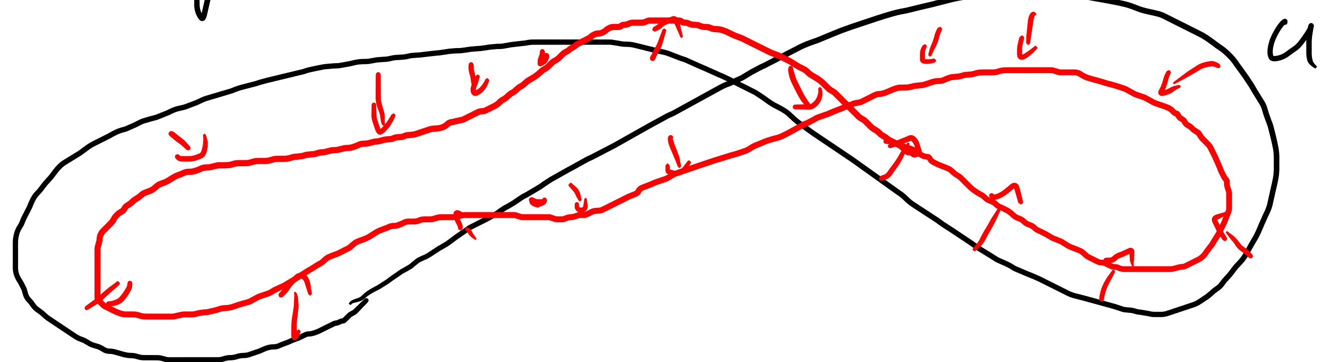
Assume this. $u : \Sigma \hookrightarrow W \Rightarrow u^*TW \cong T\Sigma \oplus \underset{\text{normal bndl}}{N_u}$

$\rightsquigarrow$ linearized CR-op. $D_u = \begin{pmatrix} D_u^T & D_u^{TN} \\ D_u^{NT} & D_u^N \end{pmatrix} \rightsquigarrow$ normal CR-op.

$D_u^N : W^{k,p}(N_u) \longrightarrow W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$.

Lemma 2: For u immersed, u is regular iff D_u^N is surj. &

$\exists$ natural iso. $T_u \mathcal{M}(J) = \ker D_u^N$.



note: $\operatorname{ind} D_u^N = (n-1)\chi(\Sigma) + 2c_1(N_u)$

$= (n-3)\chi(\Sigma) + 2[\underbrace{\chi(\Sigma) + c_1(N_u)}_{c_1(u^*TW) = c_1(A)}]$

$=$ vir-dim $\mathcal{M}_{g,0}(J,A)$.

looking for $v = \exp_u \eta$ for $\eta \in \Gamma(N_u)$ s.t. $\operatorname{im} dv(z)$ J-inv't $\forall z \in \Sigma$.

So for orientation, suff. to orient the determinant lines of D_u^N $\forall u \in \mathcal{M}(J)$.

Let $D^{\mathbb{C}} :=$ $\mathbb{C}$-linear part of D_u^N, $\quad D_s := s D_u^N + (1-s) D^{\mathbb{C}}$, $\quad 0 \le s \le 1$.

$E := N_u$, $\quad F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$, $\quad D_s : W^{k,p}(E) \to W^{k-1,p}(F)$ CR-type.

<u>Lemma 3</u>: If $n=2$, $g=0$, D_s is surjective $\forall\, s \in [0,1]$.

<u>pf</u>: E is cpx line bndl over S^2, $\operatorname{ind} D_s = 0 \quad \forall s \Rightarrow$

suff. to prove all injective. $\quad 0 = \operatorname{ind} D_s = \chi(S^2) + 2c_1(E)$

$\Rightarrow c_1(E) = -1$. $\quad$ If $\eta \neq 0 \in \ker D_s$, similarity princ. $\Rightarrow$

$$\underbrace{\# \eta^{-1}(0)}_{= c_1(E)} \geq 0, \text{ contra!}$$

$\square$

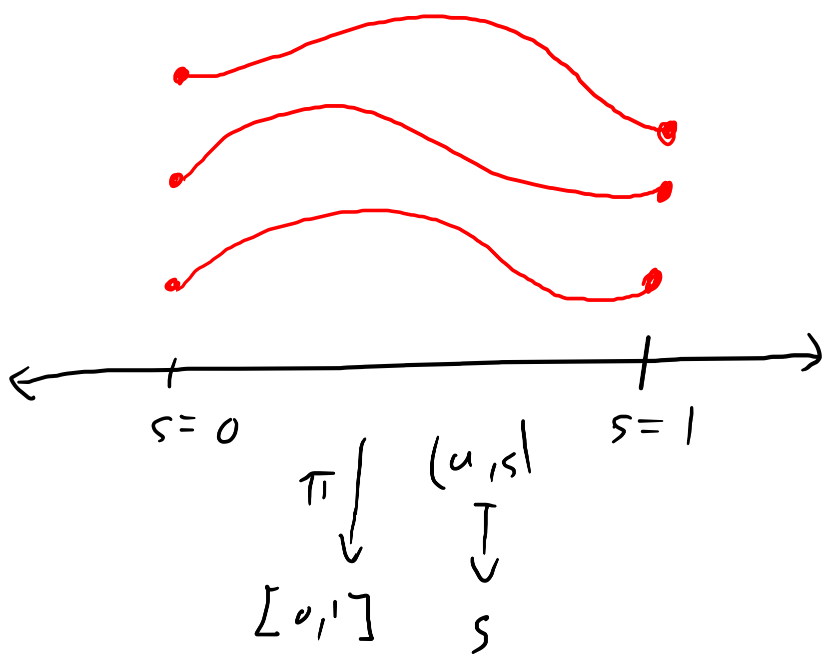
$\Rightarrow$ the canonical cpx orientation of $D^{\mathbb{C}}$ (which matches the pos. orientation of $\mathbb{R}$)

determines the pos. orientation of $\det(D_u^N) = \mathbb{R}$.

<u>false for $g > 0$</u>: no contradiction by counting zeroes

<u>false for $n > 2$</u>: $\operatorname{rk} E > 1 \Rightarrow$ cannot count zeroes of $\eta \in \ker D_s$.

<u>sanity</u> <u>check</u>: why $\mathcal{M}_{g,0}(J_0, A)$ & $\mathcal{M}_{g,0}(J_1, A)$ have same # curves:



Our pf shows: $\forall\, (u,s) \in \mathcal{M}_{g,0}(\{J_s\}, A)$

in this setting, D_u^N <u>always</u> surj.

$\Rightarrow \pi$ is a <u>submersion</u>.