EXERCISES FOR WEEK 1 (22.04.2020)

We will use the moodle forum at

`https://moodle.hu-berlin.de/course/view.php?id=95257`

for discussion of the exercises. If you write up a solution that you are happy with, feel to post it to the forum (you can upload a PDF file if you click on *Advanced* next to the *Post to forum* and *Cancel* buttons). You may also use the forum to comment on or ask questions about someone else's solution, or post a partial solution with details you are unsure about, or just to ask questions about the exercises.

1. Suppose (W, ω) is a symplectic manifold and $M \subset W$ is a closed oriented hypersurface. Denote the restriction of ω to M by $\omega_M := \omega|_{TM} \in \Omega^2(M)$.

   (a) Show that there exists a nonempty and convex set of 1-forms $\lambda \in \Omega^1(M)$ satisfying

   $$\lambda \wedge \omega_M^{n-1} > 0 \text{ everywhere on } M.$$

   Hint: How must λ behave on the characteristic line field of M?

   (b) Show that for any choice of 1-form λ as in part (a), M admits a tubular neighborhood $(-\epsilon, \epsilon) \times M \subset W$ on which ω takes the form

   $$\omega = \omega_M + d(r\lambda) \quad \text{on} \quad (-\epsilon, \epsilon) \times M,$$

   where r denotes the canonical coordinate on $(-\epsilon, \epsilon)$.
   Hint: If you flow from M along an intelligently chosen vector field transverse to M, you'll get a neighborhood on which ω matches $\omega_0 := \omega_M + d(r\lambda)$ at M, though not necessarily in a whole neighborhood of M. Now use the Moser deformation trick to find a diffeomorphism φ between neighborhoods of M that fixes M and satisfies $\varphi^*\omega = \omega_0$.

2. Let's see if you've properly internalized the Moser deformation trick yet. The classical *Morse lemma* can be interpreted as saying that if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function of the form $f(x) = f(0) + Q(x) + R(x)$ for some *nondegenerate* quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ and $R(x) = O(|x|^3)$, then there exists a diffeomorphism φ between neighborhoods of $0 \in \mathbb{R}^n$, fixing the origin, such that[1]

   $$f(\varphi(x)) = f(0) + Q(x).$$

   Use the Moser deformation trick to prove this statement, at least if the diffeomorphism φ is allowed to be of class C^1 (but not necessarily smoother).

[1] The usual form of the Morse lemma follows from this since a nondegenerate quadratic form can always be diagonalized to put it in the form $Q(x) = \sum_{j=1}^{k} x_j^2 - \sum_{j=k+1}^{n} x_j^2$ for some k.