bifurcation theory

<u>Q</u>: $\{J_\tau\}_{\tau \in \mathbb{R}}$ generic family of a.c.s's, regular $\forall\, \tau \in [\tau_0 - \varepsilon, \tau_0 + \varepsilon]$ except at $\tau = \tau_0$. How are the topologies of $\mathcal{M}(J_{\tau_0+\varepsilon})$, $\mathcal{M}(J_{\tau_0-\varepsilon})$ related?

What does $\mathcal{M}(\{J_\tau\}) := \{(u,\tau) \mid u \in \mathcal{M}(J_\tau)\}$ look like near $\tau = \tau_0$?

<u>fin.-dim. toy model</u>: $E \to M$ orbibundle (V.B./group action w. finite stabilizers)

$M \ni x \rightsquigarrow$ isotropy group G_x (finite)

$\{\sigma_\tau\}_{\tau \in \mathbb{R}}$ smooth family of sections $\sigma_\tau : M \to E$.

Let $I := \dim M - \operatorname{rk} E =: \operatorname{ind}(x)$ for $x \in \sigma^{-1}(0) =: \mathcal{M}(\sigma) \subseteq M$

$\mathcal{M} := \{(x,\tau) \in M \times \mathbb{R} \mid \sigma_\tau(x) = 0\}$, $\quad \mathcal{M}_\tau := \mathcal{M}(\sigma_\tau)$

Call $x \in \mathcal{M}_\tau$ <u>regular</u> if $D_{(x,\tau)} := D\sigma_\tau(x) : T_x M \to E_x$ is surjective.

<u>parametrically regular</u> if $L_{(x,\tau)} : T_x M \oplus \mathbb{R} \to E_x : (X,t) \mapsto D_{(x,\tau)} X + \partial_\tau \sigma(x)$ is surjective

<u>thm 1</u> (Sard-Smale):

$\forall$ finite grps. G & generic $\{\sigma_\tau\}_{\tau \in \mathbb{R}}$,

$\mathcal{M}^G := \{(x,\tau) \in \mathcal{M} \mid G_x \cong G\}$ is a smooth mfd & $k, c \in \mathbb{N}$,

$\mathcal{M}^G(k,c) := \{(x,\tau) \in \mathcal{M}^G \mid \dim \ker D_{(x,\tau)} = k,\ \dim \operatorname{coker} D_{(x,\tau)} = c\}$ is a smooth submfd whose codim. near $(x,\tau) \in \mathcal{M}^G(k,c)$ is $\dim \operatorname{Hom}_G(\ker D_{(x,\tau)}, \operatorname{coker} D_{(x,\tau)})$.

Case $G = \{1\}$: Write $\mathcal{M}^* := \mathcal{M}^G$. ind $D_{(x,\tau)} = I$, $L_{(x,\tau)}$ surj.

$\Rightarrow \mathcal{M}^*$ is a mfd of dim. $I+1$. $\mathcal{M}^*(k,c) = \emptyset$ unless $k - c = I$.

For $k = c + I$, $\dim \mathcal{M}^*(k,c) = I + 1 - \dim \mathrm{Hom}(\ker D_{(x,\tau)}, \operatorname{coker} D_{(x,\tau)})$

$$= I + 1 - c(c+I) = \begin{cases} 0 & \text{if } c = 1 \\ <0 & \text{if } c \geq 2 \end{cases}$$

$\Rightarrow u = (x,\tau) \in \mathcal{M}^*$ is regular except for a discrete subset $\mathcal{M}^*_{crit}$ characterized by $\dim \operatorname{coker} D_u = 1$.

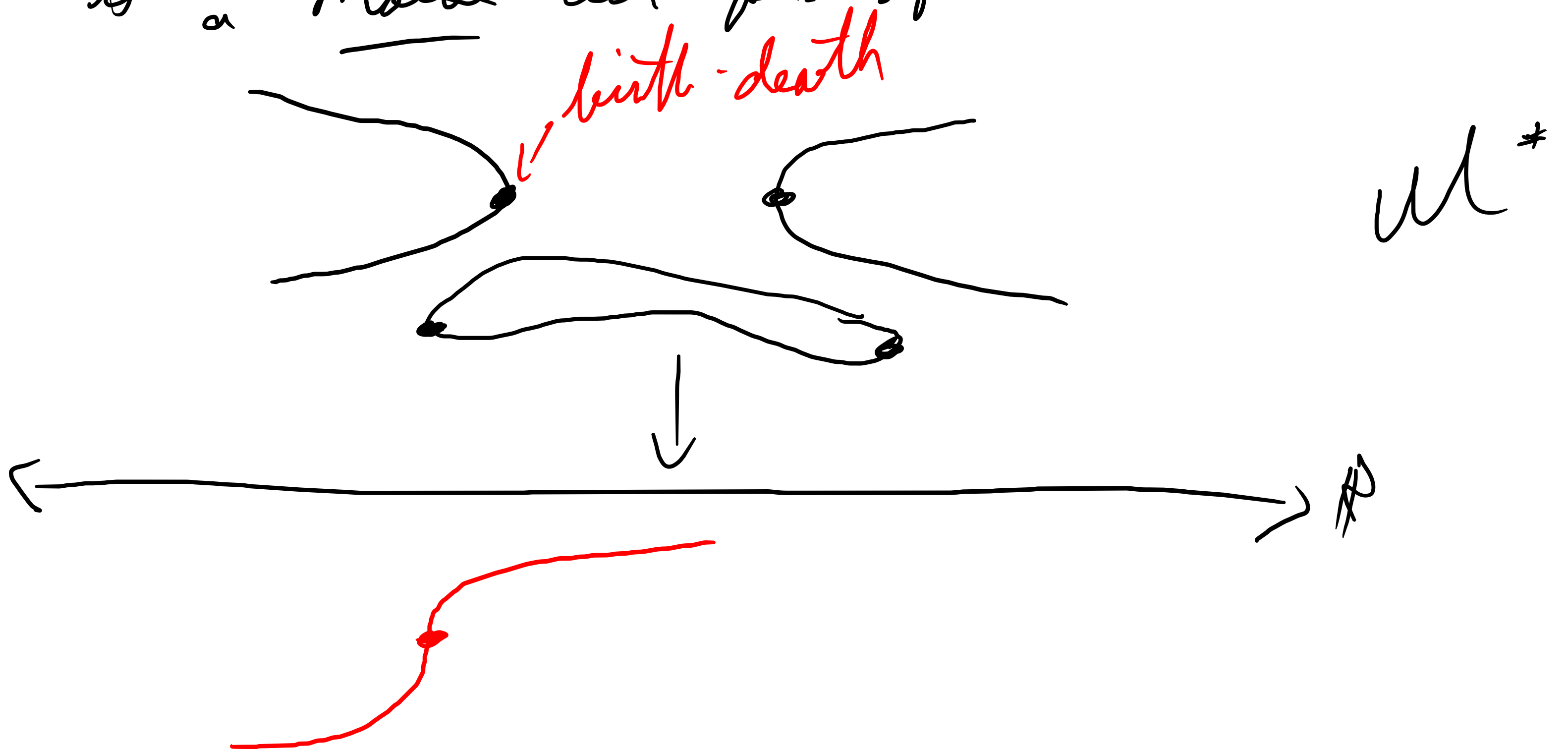
Q: Structure of $\mathcal{M}^*$ near $u_0 = (x_0, \tau_0) \in \mathcal{M}^*_{crit}$?

Ex: $\forall\, u = (x,\tau) \in \mathcal{M}^*$, $\exists$ well-def'd a surj. linear map $T_u\mathcal{M}^* \longrightarrow \mathrm{Hom}(\ker D_u, \operatorname{coker} D_u)$ given by

$$X \in \begin{tikzcd} \ker D_u \arrow[r, hook] \arrow[rrr, bend right, "D_X D_u"'] & T_x M \arrow[r, "\nabla_X D_u"] & E_x \arrow[r, "\text{proj.}"] & \operatorname{coker} D_u \end{tikzcd}$$

thm 2: Each $u_0 \in \mathcal{M}^*_{crit}$ is a Morse crit. pt. of $\mathcal{M}^* \xrightarrow{\tau} \mathbb{R} : (x,\tau) \mapsto \tau$.

ex: $I = 0$, $\dim \mathcal{M}^* = 1$.



pf: Consider a path $u_s = (x_s, \tau_s) \in \mathcal{M}^*$ through $u_0 \in \mathcal{M}^*_{\text{crit}}$,

choose vec. fld. along path: $X(s) = (X_M(s), X_{\mathbb{R}}) \in T_{u_s}\mathcal{M}^* \subseteq T_{x_s}M \oplus \mathbb{R}$.

$$L_{u_s}(X(s)) = 0 = D_{u_s}(X_M(s)) + X_{\mathbb{R}}(s)\, \partial_\tau \sigma_{\tau_s}(x_s).$$

For generic $\{\sigma_\tau\}$, can assume $\partial_\tau \sigma_{\tau_0}(x_0) \neq 0 \quad \forall\ (x_0, \tau_0) \in \mathcal{M}^*_{\text{crit}}$.

$\dim \mathcal{M}^* = I + 1 = \dim \ker D_{u_0} \Rightarrow T_{u_0}\mathcal{M}^* = \ker D_{u_0} \oplus \{0\} \subseteq T_{x_0}M \oplus \mathbb{R}$.

$\Rightarrow$ For $Y := \partial_s u_s|_{s=0} \in \ker D_{u_0} \oplus \{0\}$,

$$0 = \nabla_s \left(L_{u_s}(X(s))\right)\Big|_{s=0} = \underbrace{\left(\nabla_s D_{u_s}\Big|_{s=0}\right)\underbrace{(X_M(0))}_{\in \ker D_{u_0}}}_{\nabla_Y D_u(X_u)\ \notin\ \operatorname{im} D_0 \text{ if } D_Y D_u(X_M) \neq 0} + \underbrace{D_{u_0}\left(\nabla_s X_M(s)\Big|_{s=0}\right)}_{\operatorname{im} D_{u_0}}$$

$$+ \left(\partial_s X_{\mathbb{R}}\Big|_{s=0}\right) \cdot \partial_\tau \sigma_{\tau_0}(x_0)$$

$\Rightarrow \underbrace{\partial_s X_{\mathbb{R}}\Big|_{s=0}}_{\operatorname{Hess}_\tau(Y, X)} \neq 0.$ Given Y, can choose X to make $D_Y D_{u_0}(X) \neq 0 \Rightarrow \operatorname{Hess}_\tau(Y, X) \neq 0.$ $\blacksquare$

case $G = \mathbb{Z}_2$, $\dim \mathcal{M}^{\mathbb{Z}_2}_\tau = 0$, so $\dim \mathcal{M}^{\mathbb{Z}_2} = 1$

Arbitrary: thm 1 $\not\Rightarrow$ all $u \in \mathcal{M}^{\mathbb{Z}_2}$ are par. reg., i.e. $\mathcal{M}^{\mathbb{Z}_2} \subset \mathcal{M}$ is a mfd, but $\mathcal{M}$ might not be a mfd/orbifold near pts. $\mathcal{M}^{\mathbb{Z}_2}$.

$u = (x,\tau) \in \mathcal{M}^{\mathbb{Z}_2} \Rightarrow D_u : T_x M \to E_x$ is $\mathbb{Z}_2$-equivar., $\Rightarrow$ splits $D_u = D_u^+ \oplus D_u^-$ wrt. the trivial/nontriv. irreps of $\mathbb{Z}_2$.

$\mathcal{M}^{\mathbb{Z}_2}_{crit} = \mathcal{M}^{\mathbb{Z}_2}_+ \sqcup \mathcal{M}^{\mathbb{Z}_2}_-$, $\quad \mathcal{M}^{\mathbb{Z}_2}_\pm := \{ u \in \mathcal{M}^{\mathbb{Z}_2} \mid \dim \operatorname{coker} D_u^\pm = 1,\ D_u^\mp \text{ surj.} \}$

EX: $\mathcal{M}^{\mathbb{Z}_2}$ has birth-death at $\mathcal{M}^{\mathbb{Z}_2}_+$.

case $u_0 \in \mathcal{M}^{\mathbb{Z}_2}_-$: $\exists$ smooth 1-param fam. $u_s = (x_s, \tau_s) \subset \mathcal{M}^{\mathbb{Z}_2}$ through u_0,

$\partial_s u_s =: (X_M(s), X_{\mathbb{R}}(s)) \in \mathcal{M}^{\mathbb{Z}_2}$ satisfy $D_{u_s}(X_M(s)) + X_{\mathbb{R}}(s) \underbrace{\partial_\tau \sigma_{\tau_s}(x_s)}_{\substack{\text{generically } \neq 0 \text{ at } s=0 \\ \text{equivariance} \Rightarrow\, \in E^+_{x_0}}} = 0.$

$\Rightarrow D_{u_0}(X_M(0)) \in E^+_{x_0} \Rightarrow$ $X_{\mathbb{R}}(0) \neq 0$.

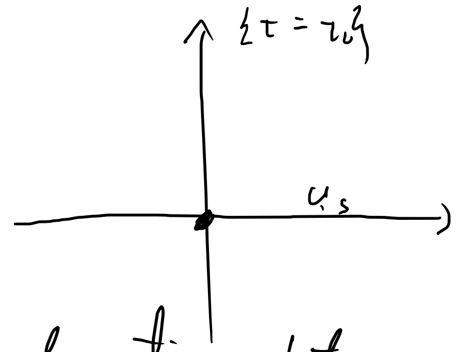
WLOG can reparametrize u_s s.t. $\tau_s = \tau_0 + s$.

obstruction bndl: Choose a 1-dim $\mathbb{Z}_2$-inv't subbndl $C \subseteq E$ near x_0 s.t. $E^-_{x_0} = \operatorname{im} D^-_{u_0} \oplus C_{x_0}$, i.e. $C_{x_0} \cong \operatorname{coker} D_{u_0}$.

EX (IFT): $\widehat{\mathcal{M}} := \{ (x,\tau) \in M \times \mathbb{R} \text{ near } (x_0, \tau_0) \mid \sigma_\tau(x) \in C_x \}$ is a smooth $\mathbb{Z}_2$-inv't submfd of dim. 2 w/ $T_{u_0} \widehat{\mathcal{M}} = \ker L_{u_0}$.

Now $\exists$ obstruction section $ob : \widehat{\mathcal{M}} \to C : (x,\tau) \mapsto \sigma_\tau(x)$ s.t. $ob^{-1}(0)$ = nbhd of u_0 in $\mathcal{M}$.

Use coords (s,t) on $\widehat{\mathcal{M}}$ s.t. $(s,0) = u_s$, $\{s=0\} = \{\tau = \tau_0\}$



so $(0,0) = u_0$ & at this pt., ∂_t spans $\ker D_{u_0}$.
$\mathbb{Z}_2$-action fixes $\{t = 0\}$, flips $\partial_t \Rightarrow$ can assume acts by $(s,t) \mapsto (s,-t)$.

Claim: $\partial_s \partial_t\, ob(0,0) \neq 0$. Pf: This is $D_{\partial_s} D_{u_0}(\partial_t)$. ∎

observation: 1st nonvanishing (say kth-order) term in Taylor series of $t \mapsto ob(0,t)$ at $t = 0$ is $\mathbb{Z}_2$-equivar. $\mathrm{Sym}(\mathbb{R}^{\otimes k}) \to \mathbb{R}$ w/ $\mathbb{Z}_2$ acts on $\mathbb{R}$ as the nontriv. repr. $\Rightarrow$ k is odd!

Generically, can arrange $k = 3$.

$ob(s,t) = a s t + b t^3 + \text{higher order} \quad (a, b \neq 0)$

WLOG $a = 1$, $b = \pm 1 \Rightarrow ob(s,t) = st + t^3 + \text{stuff}$

$= t(s \pm t^2)$