Applications

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- $\Lambda \subset Y$ single Legendrian sphere.
- $\mathcal{C}$ set of Reeb chords of Λ including the empty Reeb chord e
- $LHA(\Lambda) :=$ words of Reeb chords in $\mathcal{C}$.

$$d_{LHA}c := \sum_{|c|=|b_j|+1} n_{c;b_1...b_m} b_1 ... b_m$$

with

$$n_{c;b_1...b_m} := \#\mathcal{M}^Y_\Lambda(c_1; b_1...b_m)/\mathbb{R} \in \mathbb{Z}.$$

$LHA(\Lambda)$ is a unital algebra with 1 corresponding to the empty Reeb chord e.

Main Theorem

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Let X be the Weinstein domain resulting from attaching a handle to X_0 along Λ. If there exists a Reeb chord $c \in LHA(\Lambda)$ such that $d_{LHA}c = 1$, then

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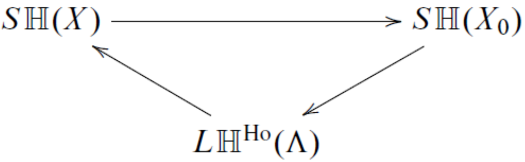
Recall the main result of the paper:



Definition of $M(\Lambda)$

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Let $M(\Lambda)$ be a left-right $LHA(\Lambda)$-module generated by

1. hat-decorated Reeb chords $\hat{c}$ with $c \in \mathcal{C}$, $|\hat{c}| = |c| + 1$
2. x auxiliary variable with $|x| = 0$.

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2. $d_M \hat{c} := xc - cx - S(d_M c)$ where $S : LHO^+(\Lambda) \longrightarrow \widehat{LHO}^+(\Lambda)$ as in the definition of LH^{Ho}:

$$S(c_1 \dots c_k) := \hat{c}_1 c_2 \dots c_k + (-1)^{|c_1|} c_1 \hat{c}_2 \dots c_k + \dots + (-1)^{|c_1 \dots c_{k-1}|} c_1 \dots \hat{c}_k.$$

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3. $d_M x = 0$.

$M^{cyc}(\Lambda)$

$M^{cyc}(\Lambda) := M(\Lambda)/_\sim$ with

$$c_1 \dots c_m \hat{a} b_1 \dots b_k \sim (-1)^{|c|\cdot|\hat{a}b|} \hat{a} b_1 \dots b_k c_1 \dots c_m,$$

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Lemma

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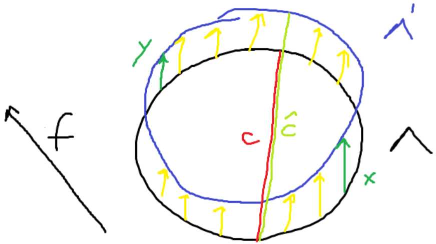
3. *The isomorphism is given by* $x \longmapsto \tau$, $c_1 \ldots c_j x c_{j+1} \ldots c_m \longmapsto c_1 \ldots c_{j-1} \check{c}_j c_{j+1} \ldots c_m$ *and* $c_1 \ldots c_{j-1} \hat{c}_j c_{j+1} \ldots c_m \longmapsto c_1 \ldots c_{j-1} \hat{c}_j c_{j+1} \ldots c_m$.

Pushing Λ

Obtain Λ' by pushing Λ along Reeb flow:



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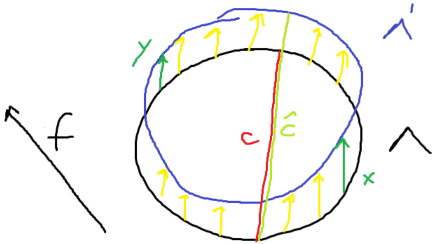

f Morse function on Λ. Reeb chords from Λ to Λ': $\forall$ chords c of Λ a $\hat{c}$, x and y corresponding to minimum and maximum of f.

$FH(L, L')$

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Definition

$FH(L, L')$ left $LHA(\Lambda)$-module and right $LHA(\Lambda')$-module generated by:

1. mixed Reeb chords starting on Λ and ending on Λ'.
2. $z = L \cap L'$.

d_{FH}

Define differential d_{FH} by:

1. On mixed Reeb chords $\hat{c}$: d_{FH} counts holmophic disks in the symplectisation of Λ with boundary in $\Lambda \cup \Lambda'$, one positive pucture at $\hat{c}$ and one negative puncture.

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It holds:

$$d_{FH} z = y.$$

Proof of Main Theorem

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Lemma $\implies L\mathbb{H}^{Ho}$ is trivial.
Surgery exact triangle $\implies$

$$S\mathbb{H}(X) \cong S\mathbb{H}(X_0).$$

□

Constructing exotic Weinstein structures on T^*S^n

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Idea: Find such a Λ with a Reeb chord c and $d_{LHA}c = 1 \implies$ symplectic homology vanishes.

Example

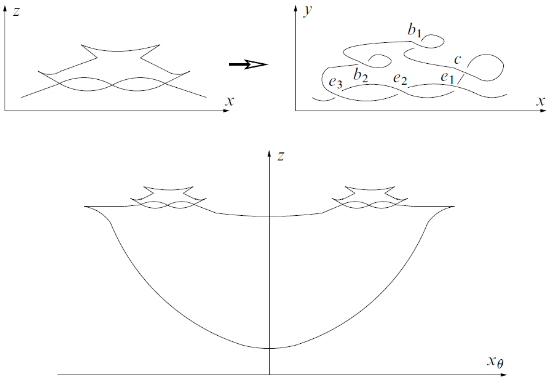

Figure 9. The Legendrian sphere Λ_T. The lower picture shows the front of Λ_T by showing its intersection with any 2–plane spanned by a unit vector $\theta \in \mathbb{R}^{n-1}$ and a unit vector in the z–direction. The upper two pictures indicates how Reeb chords arise.

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- Gradings:

$$|a| = |b_k^{max}| = |c^{max}| = n - 1$$
$$|e_j^{max}| = n - 2$$
$$|b_k^{min}| = |c^{min}| = 1$$
$$|e_j^{min}| = 0.$$

$db = 1$

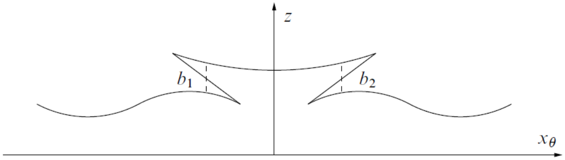

Figure 11. Rigid flow trees giving $db_k^{\min} = 1$

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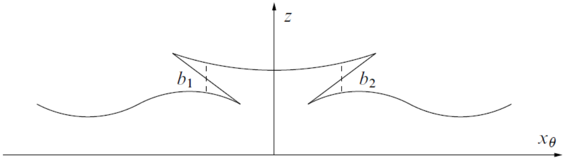

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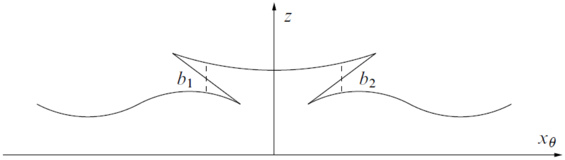

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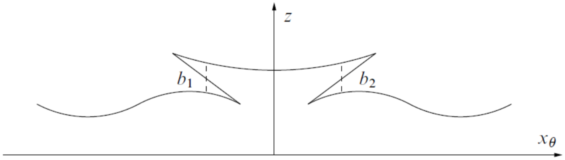

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Result by Ekholm: $d_{LHA} b_k^{min} = 1$.
$\Longrightarrow$ Attaching a sphere to Λ_T constructs an exotic Weinstein structure on T^*S^n.