Effekt of Legendrian surgery

(Bourgeois, Ekholm, Eliashberg)

$W_0^{2n} \subset W^{2n}$ Liouville domains

$V := W \setminus W_0 \cong D^n \times D^n$ crit. Weinstein handle

$\Lambda \subset M_0 := \partial W_0$ attaching sphere (assumed to be in reg. position)

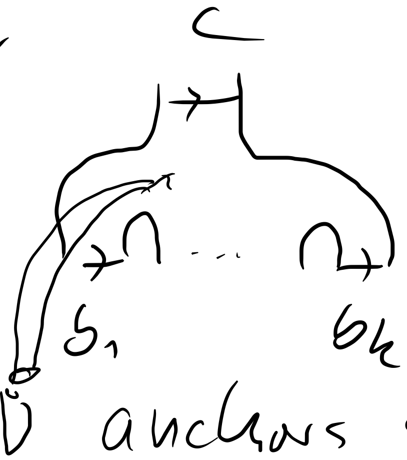
Rmk: In paper V is Weinstein cobordism and Λ is a collection of attaching spheres.

$C_* = \mathbb{k}\langle \mathcal{C} \rangle$, $\mathbb{k}$-space gen. by Reeb chords

$\mathcal{C}$ = set of Reeb chords starting and ending on Λ

$$C_*(\Lambda) := \bigoplus_{k=0}^{\infty} (\mathbb{k}\langle \mathcal{C} \rangle)^{\otimes k} \cong \mathbb{k}\langle \text{words in } \mathcal{C} \rangle$$

$\partial : C_*(\Lambda) \to C_{*-1}(\Lambda)$ bdary operator

$\partial c = \sum \#_{alg} \{$ 

in $(\mathbb{R} \times M_0, \mathbb{R} \times \Lambda)$ $b_1 \ldots b_k$

anchors allowed in W_0

ext'd ∂ via graded Leibniz rule.

def $LHO^+_*(\Lambda) := \bigoplus_{k=1}^{\infty} \mathbb{k}\langle \mathcal{C} \rangle^{\otimes k}$

and $LH^{Ho}_*(\Lambda) := \mathbb{k} \oplus \check{LHO}^+_*(\Lambda) \oplus \widehat{LHO}^+_*(\Lambda)$

where $\check{LHO}^+_*(\Lambda) = LHO^+_*(\Lambda)$ with

gen. $\check{w} = \check{c}_1 c_2 \dots c_k$

and $\widehat{LHO}^+_*(\Lambda) = LHO^+_*(\Lambda)[1]$ with

gen. $\hat{w} = \hat{c}_1 c_2 \dots c_k$

and bdry operator $\partial^{Ho} : LH^{Ho}_*(\Lambda) \circlearrowleft [-1]$

$$\begin{pmatrix} 0 & \delta & 0 \\ 0 & \check{\partial} & \varphi \\ 0 & 0 & \hat{\partial} \end{pmatrix}$$

with $\check{\partial} = \partial$ under identification $\check{LHO}^+_* = LHO^+_*$

and $\hat{\partial}(\hat{c}_1 \dots c_m) = S(\partial c_1) c_2 \dots c_m + \dots + (-1)^{|c_1|+1} \hat{c}_1 d(c_2 \dots c_m)$

$$S(b_1 \dots b_\ell) = \hat{b}_1 b_2 \dots b_\ell + (-1)^{|b_1|} b_1 \hat{b}_2 b_3 \dots b_\ell + \dots (-1)^{|b_1 \dots b_{\ell-1}|} b_1 \dots b_{\ell-1} \hat{b}_\ell$$

$\delta(\check{c}_1 c_2 \dots c_m) = 0 \quad m \geq 2$

$\delta(\check{c}_1) =$ nbrs of anchored hol. disks asympt. to c_1 at $+\infty$

$$\varphi(\hat{c}_1 c_2 \dots c_m) = \check{c}_1 c_2 \dots c_m - c_1 c_2 \dots \check{c}_m$$

Thm A: $\partial^{Ho} \circ \partial^{Ho} = 0$ and

$$LH^{Ho}_*(\Lambda) := \frac{\ker \partial^{Ho}}{\operatorname{im} \partial^{Ho}}$$

indep. of choices and legendrian isotopy of Λ.

Thm B: (Thm 5.6) There is an exact triangle

$$\begin{tikzcd} SH_*(W) \arrow[rr, "\text{Vit.}", "\text{transfer}"'] & & SH_*(W_0) \arrow[dl] \\ & LH^{Ho}_*(\Lambda) \arrow[ul] & \end{tikzcd}$$

Rmk: (Thm 5.4) There is also a version

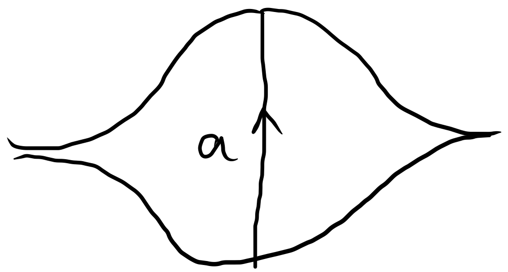
$$\begin{tikzcd} SH^+_*(W) \arrow[rr] & & SH^+(W_0) \arrow[dl] \\ & LH^{Ho+}_*(\Lambda) \arrow[ul] & \end{tikzcd}$$

§2 Example

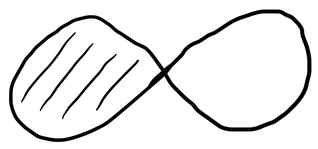
take $\Lambda \subset \mathbb{R}^{2n-1} \overset{\text{open}}{\subset} S^{2n-1} = \partial B^{2n}$

$W_0 = B^{2n}$ $\quad \mathbb{k} = \mathbb{Q}$ $\quad$ Leg. unknot

front proj:



Lagr.:



one chord a $\quad |a| = n-1$ $\quad da = 0$

LHO	1	a	a^2	a^3	....
degr.	0	$n-1$	$2(n-1)$	$3(n-1)$	....

gen. of LH^{Ho}	degree
1	0
$\check{a}$	$n-1$
$\hat{a}$	n
$\check{a}^2$	$2(n-1)$
$\hat{a}^2$	$2(n-1)+1$
$\check{a}^3$	$3(n-1)$
$\hat{a}^3$	$3(n-1)+1$
⋮	

One can verify that boundary operator vanishes up to case n even and then

$$\partial(\hat{a}^{2\ell}) = \check{a}^{2\ell} \quad \forall \ell \in \mathbb{N}$$

ThmB $\Rightarrow$

$$SH_*(W) = \begin{cases} \mathbb{Q} & * = 0,\ j(n-1) \\ & \quad j(n-1)+1 \\ & \quad j = 1,2,\ldots \\ 0 & \text{else} \end{cases}$$

n odd.

n even:

$$SH_*(W) = \begin{cases} \mathbb{Q} & * = 0,\ j(n-1),\ j(n-1)+1 \\ & \quad j = 1,3,5,\ldots \\ 0 & \text{else} \end{cases}$$

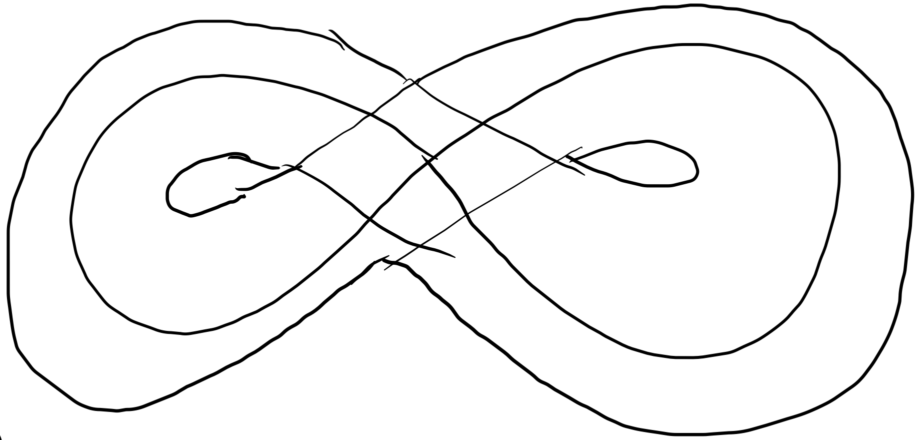
<u>know</u>: $W \simeq$ unit discbundle in T^*S^n

§3 Dynamical change

$M_0 = \partial W_0 \rightsquigarrow M = \partial W$

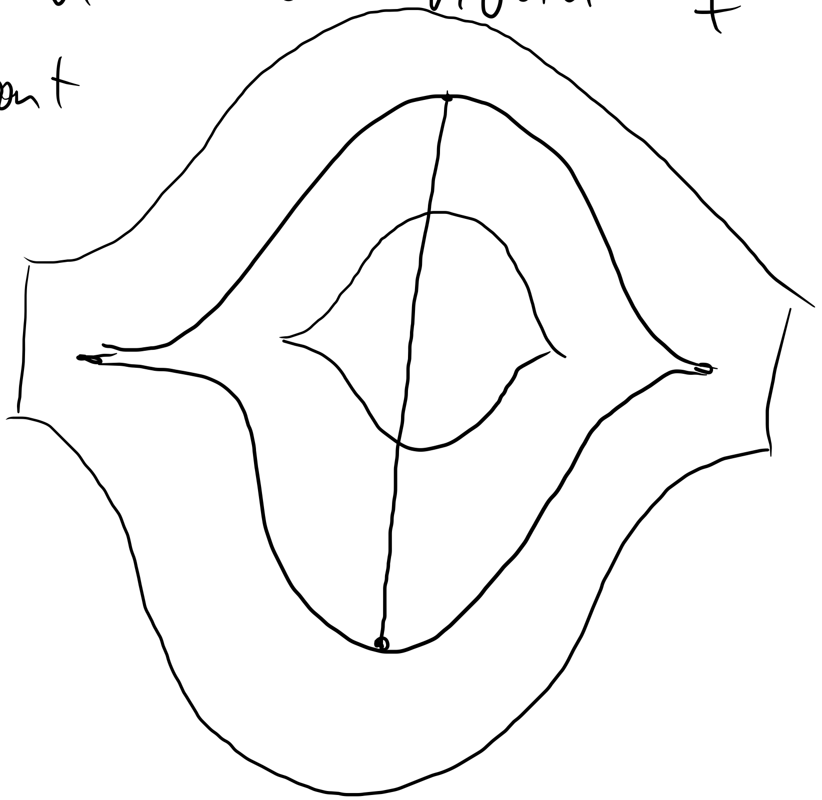
$= M_0 \setminus U \cup_{\partial U} \left(B^n \times S^{n-1}\right)$

Lagr. proj



$U \subset M_0$ nbhd of Λ

front



V_+

U

V_-

sympl. leaves $V_\pm := \{z = \pm\varepsilon\} \cap U$

know $V_\pm \simeq$ disk bdles in T^*S^{n-1}

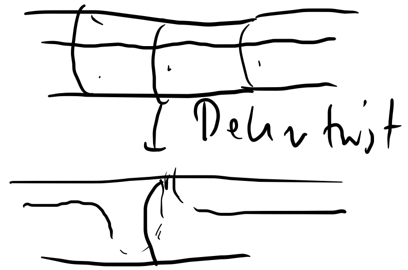
Reeb flow induces symplectomorphism

$\phi_0 : V_- \longrightarrow V_+$ Reebflow in M_0

$\phi : V_- \longrightarrow V_+$ Reebflow in M

<u>Lem:</u> $\phi_0 \circ \phi^{-1} : V_+ \circlearrowleft$ is gen. Dehn twist in T^*S^{n-1}

Assume $\Lambda \times \mathbb{R}_+ \xrightarrow[\text{flow}]{\text{Reeb}} M_0$ intersects $\Lambda \subset M_0$ transversely



<u>Key-lemma:</u> (Section 6.1) For all μ and suff. small neighbourhood U: For all periodic orbits γ in M of period $\leq \mu$ either γ never enters U or there exists a word of Reeb chords $w = c_1 c_2 \dots c_m$ st. $\operatorname{im} \gamma \subset \operatorname{nbh}(w)$

Conversely for every word $w = c_1 \dots c_m$ there exists a unique periodic orbit γ_w which passes through flow tubes of $c_1, c_2, \dots, c_m$ exactly once.

Proof Theorem B

from hom. alg: If $\varphi: C_* \to D_*$ is a chain map between chain cx. C_* and D_*. Consider chain cx:

$$\mathrm{cone}_* \varphi = C_* \oplus D_{*-1}$$

with bdary operator $\begin{pmatrix} \partial^C_* & 0 \\ (-1)^* \varphi & \partial^D_* \end{pmatrix}$

get ex. triangle

$$\begin{tikzcd} H.(C_*) \arrow[rr, "H(\varphi)"] & & H(D_*) \arrow[dl] \\ & H(\mathrm{cone}\,\varphi) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Idea: use this with a chain map

$$SC_*(W_0) \xrightarrow{\varphi} LH_*^{HO}(\Lambda)$$

Viterbo transfer:

$$\begin{array}{ccc} T: SC_*(W) & \longrightarrow & SC_*(W_0) \\ \| & & \| \\ C_*(W) \oplus \check{CH}_*(W) \oplus \hat{CH}_*(W) & & C_*(W_0) \oplus \check{CH}(W_0) \oplus \hat{CH}(W_0) \end{array}$$

where $T = \begin{pmatrix} \psi_{00} & \psi_{01} & 0 \\ 0 & \psi_{11} & \psi_{12} \\ 0 & 0 & \psi_{22} \end{pmatrix}$

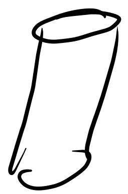
ψ_{00} = projection

$\left(C_*(W) = C_*(W_0) \oplus k\langle p_0 \rangle \right)$

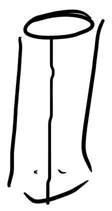
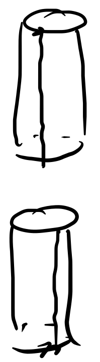
$$\psi_{11}(\check{\gamma}) = \sum_{|\beta|=|\gamma|} \frac{m(\gamma,\beta)}{\kappa(\beta)} \check{\beta}$$

$$\psi_{12}(\check{\gamma}) = \sum_{|\beta|=|\gamma|-1} \check{m}(\gamma,\beta) \hat{\beta}$$

$$\psi_{22}(\hat{\gamma}) = \sum_{|\beta|=|\gamma|} \frac{m(\gamma,\beta)}{\kappa(\gamma)} \hat{\beta}$$

where $m(\gamma,\beta) = \#\Big\{$ γ  β in $\hat{V}$ $\Big\}$

$\check{m}(\gamma,\beta)$

$= \#\Big\{$ γ  β in $\hat{V}$ $\Big\}$ markers match $+ \#\Big\{$ γ  γ' β in $\hat{W}$ in $\hat{V}$ $\Big\}$ $p_\gamma, p_{\gamma'}, p_\beta$ cyclic order

$+ \#\Big\{$ γ  β' β in $\hat{V}$ in $\hat{W}_0$ $\Big\}$ $p_\gamma, p_{\beta'}, p_\beta$ cyclic order.

$\hat{V}$ = completion of Handle $W \setminus W_0$

$\hat{W}$ = compl. of W

$\hat{W}_0$ = compl. of W_0

define φ

$$SC_*(W_0) \longrightarrow LH^{Ho}_*(\Lambda)$$

$$\cong$$

$$C_*(W_0) \oplus \check{CH}_*(W_0) \oplus \hat{CH}_*(W_0) \quad \Vert\wr \quad k\langle p_0\rangle \oplus \check{LHO}^+_*(\Lambda) \oplus \hat{LHO}^+_*(\Lambda)$$

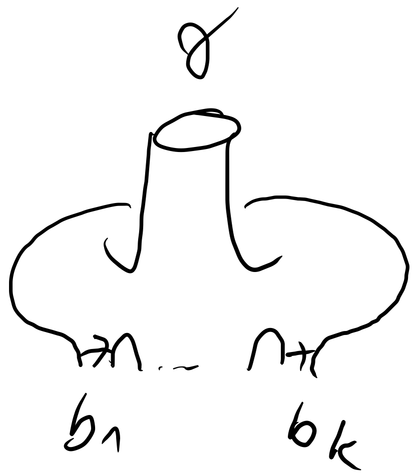
$$\varphi(\hat{\gamma}) = \sum_{|w|=|\gamma|-1}' \frac{m(\gamma,w)}{\kappa(\gamma)} S(w)$$

$$\varphi(\check{\gamma}) = \sum_{|w|=|\gamma|-1}' \check{m}(\gamma,w)\check{w} + \sum_{|w|=|\gamma|-2}' \hat{m}(\gamma,w)\hat{w} + m(\gamma)\cdot p_0$$

$$\varphi(p) = m(p,p_0)\cdot p_0$$

$m(p,p_0)$ number of Morse traj from p to p_0

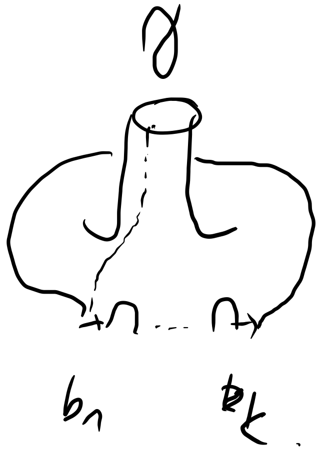
$m(\gamma,w) = \#\{$



hol. disk in $(\mathbb{R}\times M_0, \mathbb{R}\times\Lambda)\}$

$w = b_1 \dots b_k$

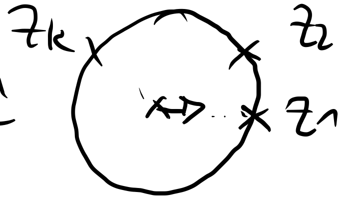
$\check{m}(\gamma,w) = \#\{$



in $(\mathbb{R}\times M_0, \mathbb{R}\times\Lambda)\}$

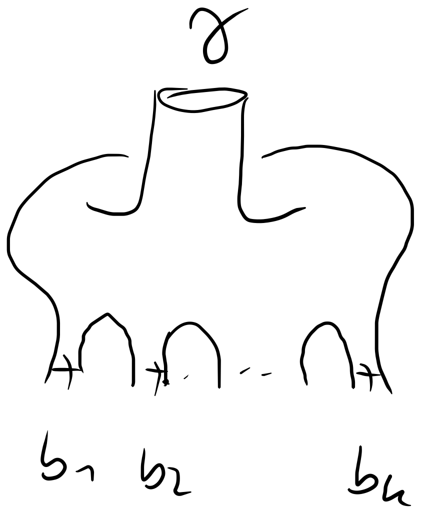
$w = b_1 \dots b_k$

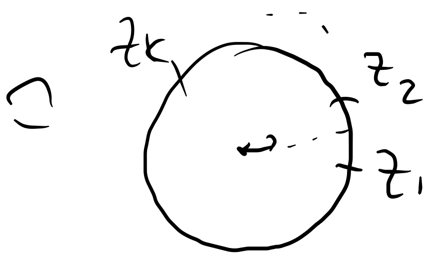
$\cong$



asympt. marker points to z_1

$\hat{n}(y, w) = \# \Big\{$

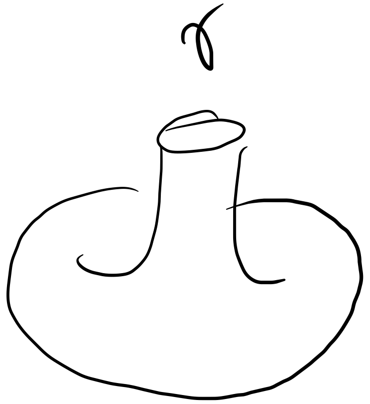




in $(\mathbb{R}\times M_0, \mathbb{R}\times\Lambda)$

asympt. marker points in arc $[z_1, z_2]$ $\Big\}$

$n(y) = \# \Big\{$



in $(\hat{V}, \hat{L})$ $\Big\}$

$L \in V$ unstable manifold of p_0

$\hat{L}$ = completion of L

goal: define a quasi-isomorphism

$$SC_*(W) \xrightarrow{F^W} \mathrm{cone}_*\varphi$$

$$SC_*(W) = C_*(W) \oplus \check{CH}_*(W) \oplus \hat{CH}_*(W)$$

$$\mathrm{cone}_*\varphi \cong C_*(W_0) \oplus \check{CH}_*(W_0) \oplus \hat{CH}_*(W_0) \oplus \Bbbk\langle p_0\rangle \oplus L\check{HO}^+_*(\Lambda) \oplus L\hat{HO}^+_*(\Lambda)$$

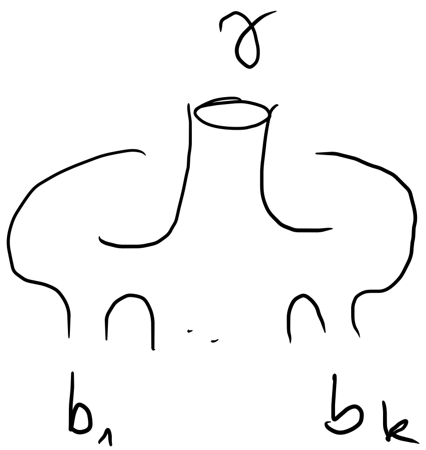
$$= C_*(W) \oplus \check{CH}_*(W_0) \oplus \hat{CH}_*(W_0) \oplus L\check{HO}^+(\Lambda) \oplus L\hat{HO}^+_*(\Lambda)$$

$$F^W(p) = p$$

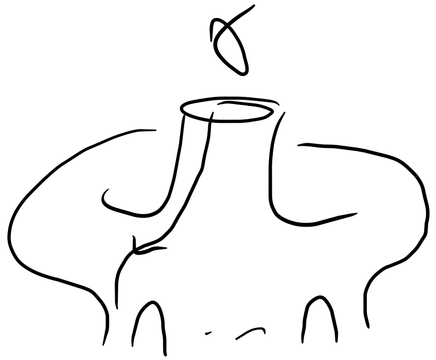
$$F^W(\check{\gamma}) = T(\check{\gamma}) + \sum_w{}' \check{m}(\gamma, w)\check{w} + \sum{}' \hat{m}(\gamma, w)\hat{w}$$

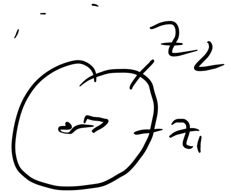
$$F^W(\hat{\gamma}) = T(\hat{\gamma}) + \sum \frac{m(\gamma, w) S(w)}{K(\gamma)}$$

where

$m(\gamma, w) = \#\{$ 

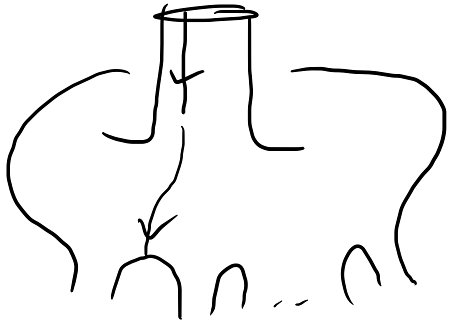
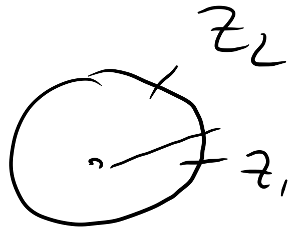
in $(\hat{V}, \hat{L})\}$

$\check{m}(\gamma, w) = \#\{$ 

$\simeq$ 

$(\hat{V}, \hat{L}\}$

asympt. marker
points to z_1

$\hat{m}(\gamma, w) = \#\{$ 
$\simeq$ 

in
$(\hat{V}, \hat{L})\}$

asympt. marker
points to arc
$[z_1, z_2]$