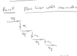
Symplectic homology III

Recall: flow lines with cascades



$SH_*(V) = \varinjlim_{H \in Ad(V)} FH(H, f, J)$

$H_+ \prec H_-$ ... $H_+ \leq H_-$ on $(R, \infty) \times \Sigma$

For action filtration $SH_*^{(a,b)}(V)$

we need $H_+ \leq H_-$ everywhere and H_t ... everywhere ...

and $H|_V < 0$, or $H|_W < 0$ for $W \subset V$ Liouville subdom.

or $H|_{\partial V}$ (this gives $RFH(V)$)

Today: Transfer map, canonical choice of Ham., Describing SH via holomorphic curves

Viterbo transfer map

Let $W \subset V$ be a Liouville subdomain, i.e. $(W, \lambda|_W)$ is a Liouville domain (in part. Y points outwards along ∂W)

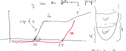
Thm (Viterbo)

There is an action filtration respecting homomorphism

$$\mathcal{T} : SH_*^{(a,b)}(V) \longrightarrow SH_*^{(a,b)}(W)$$

Proof: Consider two classes of Hamiltonians

- $H \in Ad(V)$, $H|_V < 0$, call this $\widetilde{Ad}(V)$
- $G \in Ad(V)$: $G|_W < 0$ and $G(r,p) = g(e^r)$, where g has the following profile



with $\mu' < \mu$, such that all $x \in P(G)$ with $x \subset W$ satisfy $\mathcal{A}^G(x) < 0$. Call this class $Ad(W \subset V)$.

claim 1: $\varinjlim_{\widetilde{Ad}(V)} FH^{(a,b)}(H, f, J) = SH^{(a,b)}(V)$

Proof: Constr. of $SH^{(a,b)}(V)$

claim 2: $\varinjlim_{Ad(W \subset V)} FH^{(a,b)}(G, f_G, J) = SH^{(a,b)}(W)$, if $a > -\varepsilon$ for ε small

Morse fct.

Proof:
- On $(0, \varepsilon) \times \partial W$, let J be of contact type
  $\Rightarrow$ Use the No-Escape-Lemma: Flow cylinders with both ends in W stay in W for all time!
- These flow cylinders look exactly like the contribution to the chain complex for $SH^{(a,b)}(W)$
- all orbits outside of W have negative ($< -\varepsilon$) action

□

Now, as $H \in \widetilde{Ad}(V)$, it is somewhere of the form $H(r,p) = \mu_H' \cdot e^r + \beta$, $\mu_H', \beta \in \mathbb{R}$, $\mu_H' \notin \mathrm{Spec}(\Sigma, \alpha)$

Then, we can construct $G_H \in Ad(W \subset V)$ with $\mu_G' = \mu_H'$ and $G_H \geq H$ everywhere

This gives a order preserving map $\widetilde{Ad}(V) \to Ad(W \subset V)$ and the continuation maps $\sigma(G_H, H)$ fit into the commutative diagram (for $H_+ < H_-$)

$$\begin{tikzcd}
FH_*^{(a,b)}(H_+, f_+, J_+) \arrow[r, "\sigma(G_{H_+}, H_+)"] \arrow[d, "\sigma(H_-, H_+)"'] & FH_*^{(a,b)}(G_{H_+}, f_{G_+}, J_{G_+}) \arrow[d, "\sigma(G_{H_-}, G_{H_+})"] \\
FH_*^{(a,b)}(H_-, f_-, J_-) \arrow[r, "\sigma(G_{H_-}, H_-)"] & FH_*^{(a,b)}(G_{H_-}, f_{G_-}, J_{G_-})
\end{tikzcd}$$

$\Rightarrow$ get in the direct limit a homomorphism

$$SH_*^{(a,b)}(V) = \varinjlim_{\widetilde{Ad}(V)} FH_*^{(a,b)}(H, f, J) \xrightarrow{\mathcal{T}} \varinjlim_{Ad(W \subset V)} FH_*^{(a,b)}(G, f_G, J) = SH_*^{(a,b)}(W)$$

Different approach (not for action filtration)

Just consider Ham. G

Then $SH(W) = \varinjlim FH^{(-\varepsilon, \infty)}(G, f_G, J)$

and $SH(V) = \varinjlim FH(G, f_G, J)$

and $\to SH^{(-\infty, \varepsilon)}(V) \to SH(V) \xrightarrow{\mathcal{T}} SH^{(\varepsilon, \infty)}(V) \to$ (for Ham. G with $G|_W < 0$)

Rmk:
- If V is obtained from W by add. a subcrit. handle H, then the transfer map $\mathcal{T}$ is an iso.
- What happens if H critical is subject of the BEE-paper!

Special Hamiltonians

Usually, one considers Ham. only of the form

$H(y) = \begin{cases} \cdot\ C^2\text{-small Morse fct. inside } V \\ \cdot\ h(e^r) \text{ on } (-1, \infty) \times \Sigma \\ \text{with } h'' \geq 0 \text{ and } h(e^r) = ae^r + b \text{ for } r >> 0,\ a \notin \mathrm{Spec}(\Sigma, \alpha) \\ \text{and } h''(e^r) \neq 0, \text{ if } h'(e^r) \in \mathrm{Spec}(\Sigma, \alpha) \end{cases}$

Then $P(H)$ consists of two types:
- const. orbits at crit. pts of H
- non-const. orbits in the area $(-1, \infty) \times \Sigma$

The second type corresponds to Reeb orbits of length $h'(e^r)$ on level $r = $ const., as $X_H(r,p) = h'(e^r) \cdot R_\alpha(p)$ on $(-1, \infty) \times \Sigma$.

The action of the second type of orbits (i.e. for $\dot{x} = X_H(x)$) is

$$\mathcal{A}^H(x) = \int_{S^1} x^*\lambda - H(x(t))\,dt = \int_{S^1} \lambda(\dot{x}(t)) - H(x(t))\,dt$$
$$= \int_{S^1} e^r \alpha(h'(e^r) \cdot R_\alpha) - h(x(t))\,dt = e^r \cdot h'(e^r) - h(e^r)$$

Why $h'' \geq 0$? Consider the fct. $\mathcal{A}(r) := h'(e^r) \cdot e^r - h(e^r)$

$\Rightarrow \mathcal{A}'(r) = h''(e^r) e^{2r} + h'(e^r) \cdot e^r - h'(e^r) \cdot e^r = h''(e^r) \cdot e^{2r} \geq 0$

$\Rightarrow$ Action increases in the r-direction

As ∂ decreases the action, it decreases the r-level

As $h'' \geq 0$, we find $h'(e^{r_-}) \geq h'(e^{r_+})$ for $r_- \geq r_+$

$\Rightarrow \partial$ decreases also the period/length of the underlying Reeb orbits. $\Rightarrow$ Can filter $SH(V)$ by the lengths of Reeb orbits.

Towards the defn. of $SH(V)$ via holomorphic curves

Most authors assume that α is a non-deg. contact form on Σ, meaning that the Reeb flow is trans. non-deg., i.e.

$\ker(\mathbb{1} - D\varphi_{R_\alpha}^T(\gamma(0))) = \mathrm{span}(R_\alpha(\gamma(0)))$ $\forall T$-per. Reeb orbit γ

$\Rightarrow$ Reeb orbits are isolated S^1-families via reparam. by time shift.

A perfect Morse fct. on S^1 has two critical points, Max & min

Let H be of the above form and α non-deg., then

for $x \in P(H)$ non-const., denote by γ_x the associated Reeb orbit, i.e. $x(t) = \gamma(h'(e^r) \cdot t)$, where $x(t) \in \{r\} \times \Sigma$, and let $\hat{\gamma}_x$ be the Minimum, $\check{\gamma}$ be the Maximum.

Then! $\mu(\hat{\gamma}_x) = \mu_{CZ}^\xi(\gamma) + 1$, $\mu(\check{\gamma}) = \mu_{CZ}^\xi(\gamma)$

Rem (correction from last time)

$\mu_{CZ}^\xi(\gamma)$ is the transverse Conley-Zehnder index (i.e. measured only to the contact field ξ)

Last time we had $\dim \hat{\mathcal{M}}_J(C_-, C_+) = \mu_{CZ}(C_+) - \mu_{CZ}(C_-) + \frac{\dim C_+ + \dim C_-}{2}$

Better (now true) $\dim \hat{\mathcal{M}}_J = \mu_{CZ}^\xi(C_+) - \mu_{CZ}^\xi(C_-) + \frac{\dim C_+ + \dim C_-}{2} + \frac{\mathrm{sign}(h''(C_+)) - \mathrm{sign}(h''(C_-))}{2}$

$\Rightarrow$ Index of a generator $\mu(c) = \mu_{CZ}^\xi(c) + \frac{1}{2}\,\mathrm{sign}(h''(c)) + \mathrm{ind}_f(c)$

Let us denote by $S_\gamma \subset P(H)$ the S^1-family of 1-per. orbits of X_H associated to per. Reeb orbit γ.

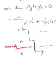
Then $\hat{\mathcal{M}}_J(S_{\gamma_-}, S_{\gamma_+}) = \mu_{CZ}^\xi(\gamma_+) - \mu_{CZ}^\xi(\gamma_-) + \frac{1+1}{2} + \frac{1-1}{2}$
$= \mu_{CZ}^\xi(\gamma_+) - \mu_{CZ}^\xi(\gamma_-) + 1$

$\Rightarrow \mathcal{M}_J(S_{\gamma_-}, S_{\gamma_+}) = \hat{\mathcal{M}}_J(S_{\gamma_-}, S_{\gamma_+})/\mathbb{R}$ is non-empty if $\mu_{CZ}^\xi(\gamma_+) \geq \mu_{CZ}^\xi(\gamma_-)$

For $y = \hat{\gamma}_+$ or $\check{\gamma}_+ \in S_{\gamma_+}$, and $x = \hat{\gamma}_-$ or $\check{\gamma}_- \in S_{\gamma_-}$ we have

$\dim \mathcal{M}_J(x, y) = \mu(y) - \mu(x) - 1 = \left(\mu_{CZ}^\xi(\gamma_+) + \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}\right) - \left(\mu_{CZ}^\xi(\gamma_-) + \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}\right) - 1$

$\Rightarrow \dim \mathcal{M}_J(x, y) = 0 \Rightarrow \mu_{CZ}^\xi(\gamma_+) - \mu_{CZ}^\xi(\gamma_-) = \begin{cases} 0 & (\hat{\ },\check{\ }) \\ 1 & (\hat{\ },\hat{\ }) \text{ or } (\check{\ },\check{\ }) \\ 2 & (\check{\ },\hat{\ }) \end{cases}$



$\mu_{CZ}^\xi(\gamma_1) - \mu_{CZ}^\xi(\gamma_0) \geq 0$, ≥ 0, ≥ 0 and $\mu_{CZ}^\xi(\gamma_3) - \mu_{CZ}^\xi(\gamma_0) \leq 2$

$\Rightarrow$ there are at most two cascades with pos. difference for μ_{CZ}

- If $y = \hat{\gamma}_+$, $x = \check{\gamma}_-$ there are no cascades with pos. diff.
- If $y = \hat{\gamma}_+$, $x = \hat{\gamma}_-$ or $y = \check{\gamma}_+$, $x = \check{\gamma}_-$ there is at most one cascade with μ_{CZ}-diff $= 1$
- If $y = \check{\gamma}_+$, $x = \hat{\gamma}_-$ there is 1 cascade with μ_{CZ}-diff. $= 2$ or 2 cascades each with μ_{CZ}-diff. $= 1$

goal: Will see that for "slow" Hamiltonians

$\mathcal{M}(S_{\gamma_-}, S_{\gamma_+}) = \emptyset$ if $\mu_{CZ}^\xi(\gamma_+) = \mu_{CZ}^\xi(\gamma_-)$