PROBLEM SET 4
Due: 25.05.2023

Instructions

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Note: This problem set is a bit longer than usual—and you have two weeks to do it—because Thursday the 18th is a holiday. It includes material from three lectures, up to Tuesday the 16th.

Problems

1. Prove that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic for any $n \geqslant 2$.
   Hint: If $\mathbb{R}$ and $\mathbb{R}^n$ are homeomorphic, then so are $\mathbb{R}\backslash\{t\}$ and $\mathbb{R}^n\backslash\{x\}$ for some $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Find a topological property that $\mathbb{R}^n\backslash\{x\}$ has and $\mathbb{R}\backslash\{t\}$ does not.

2. (∗) In lecture we proved that in any space X that is locally compact and Hausdorff, every neighborhood of every point $x \in X$ contains a compact neighborhood of x. Extend this result to prove the following: if X is locally compact and Hausdorff, then for any nested pair of subsets $K \subset \mathcal{U} \subset X$ with K compact and $\mathcal{U}$ open, there exists an open set $\mathcal{V} \subset X$ with compact closure $\bar{\mathcal{V}}$ such that $K \subset \mathcal{V} \subset \bar{\mathcal{V}} \subset \mathcal{U}$.

3. Given a space X, the collection of all connected components of X can be viewed as a collection of topological spaces $\{X_\alpha\}_{\alpha\in I}$, where each X_α is endowed with the subspace topology as a subset of X. Each therefore comes with a continuous inclusion map $i_\alpha : X_\alpha \hookrightarrow X$, and these can be assembled into a map
$$i : \coprod_{\alpha\in I} X_\alpha \to X,$$
   defined by the property that $i|_{X_\alpha} = i_\alpha$ for each $\alpha \in I$. This map is obviously a bijection, and the definition of the disjoint union topology implies that it is continuous. Show that i is a homeomorphism if and only if every X_α is an open subset of X.[1]

4. (a) Prove that if X and Y are both connected, then so is $X \times Y$.[2]
   Hint: Start by showing that for any $x \in X$ and $y \in Y$, the subsets $\{x\} \times Y$ and $X \times \{y\}$ in $X \times Y$ are connected. Then think about continuous maps $X \times Y \to \{0, 1\}$.
   (b) Show that for any collection of path-connected spaces $\{X_\alpha\}_{\alpha\in I}$, the space $\prod_{\alpha\in I} X_\alpha$ is path-connected in the usual product topology.
   Hint: You might find Problem Set 2 #3(d) helpful.
   (c) Consider $\mathbb{R}^{\mathbb{N}}$ with the "box topology" which we discussed in Problem Set 2 #5. Show that the set of all elements $f \in \mathbb{R}^{\mathbb{N}}$ represented as functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy $\lim_{n\to\infty} f(n) = 0$ is both open and closed, hence $\mathbb{R}^{\mathbb{N}}$ in the box topology is not connected (and therefore also not path-connected).

5. For each of the following spaces, determine whether it is (i) Hausdorff, (ii) locally compact, (iii) connected, (iv) locally path-connected.[3]
   (a) (∗) The irrational numbers $\mathbb{R}\backslash\mathbb{Q}$

[1] As shown in lecture, this condition holds whenever X is locally connected or has only finitely many connected components, but e.g. it does not hold for $X = \mathbb{Q}$.
[2] The analogous statement about infinite products is also true, but it takes more work to prove it. If you're curious, you'll find some hints about the latter in Munkres' *Topology* (second edition), §23, Exercise 10.
[3] You should always assume unless otherwise specified that $\mathbb{R}$ is endowed with its standard topology, and all spaces derived from it as subsets/products/quotients etc. carry the natural subspace/product/quotient topology.

(b) $\{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$

(c) The quotient group $\mathbb{R}/\mathbb{Q}$

6. There is a cheap trick to view any topological space as a compact space with a single point removed. For a space X with topology $\mathcal{T}$, let $\{\infty\}$ denote a set consisting of one element that is not in X, and define the *one point compactification* of X as the set $X^* = X \cup \{\infty\}$ with topology $\mathcal{T}^*$ consisting of all subsets in $\mathcal{T}$ plus all subsets of the form $(X \backslash K) \cup \{\infty\} \subset X^*$ where $K \subset X$ is closed and compact.

   (a) Verify that $\mathcal{T}^*$ is a topology and that X^* is always compact.

   (b) ($*$) Show that if X is first countable and Hausdorff, a sequence in $X \subset X^*$ converges to $\infty \in X^*$ if and only if it has no convergent subsequence with a limit in X. Conclude that if X is first countable and Hausdorff, X^* is sequentially compact.

   (c) Show that for $X = \mathbb{R}$, X^* is homeomorphic to S^1. *(More generally, one can use stereographic projection to show that the one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.)*

   (d) Show that if X is already compact, then X^* is homeomorphic to the disjoint union $X \amalg \{\infty\}$.

   (e) Show that X^* is Hausdorff if and only if X is both Hausdorff and locally compact.

   Notice that $\mathbb{Q}$ is not locally compact, since every neighborhood of a point $x \in \mathbb{Q}$ contains sequences without convergent subsequences, e.g. any sequence of rational numbers that converges to an irrational number sufficiently close to x. The one point compactification $\mathbb{Q}^*$ is a compact space, and by part (b) it is also sequentially compact, but those are practically the only nice things we can say about it.

   (f) Show that for any $x \in \mathbb{Q}$, every neighborhood of x in $\mathbb{Q}^*$ intersects every neighborhood of ∞, so in particular, $\mathbb{Q}^*$ is not Hausdorff.
   Advice: Do not try to argue in terms of sequences with non-unique limits (cf. part (g) below), and do not try to describe precisely what arbitrary compact subsets of $\mathbb{Q}$ can look like (the answer is not nice). One useful thing you can say about arbitrary compact subsets of $\mathbb{Q}$ is that they can never contain the intersection of $\mathbb{Q}$ with any open interval. (Why not?)

   (g) Show that every convergent sequence in $\mathbb{Q}^*$ has a unique limit. *(Since $\mathbb{Q}^*$ is not Hausdorff, this implies via a result mentioned in lecture—see Prop. 6.13 in the notes—that $\mathbb{Q}^*$ is not first countable. In particular, ∞ does not have a countable neighborhood base.)*

   (h) Find a point in $\mathbb{Q}^*$ with a neighborhood that does not contain any compact neighborhood.

7. Given spaces X and Y, let $C(X,Y)$ denote the set of all continuous maps from X to Y, and consider the natural *evaluation map*

$$\text{ev} : C(X,Y) \times X \to Y : (f,x) \mapsto f(x).$$

   It is easy to show that ev is a continuous map if we assign the discrete topology to $C(X,Y)$, but usually one can also find more interesting topologies on $C(X,Y)$ for which ev is continuous. The *compact-open topology* is defined via a subbase consisting of all subsets of the form

$$\mathcal{U}_{K,V} := \left\{ f \in C(X,Y) \;\middle|\; f(K) \subset V \right\},$$

   where K ranges over all compact subsets of X, and V ranges over all open subsets of Y.

   (a) Show that if Y is a metric space, then convergence of a sequence $f_n \in C(X,Y)$ in the compact-open topology means that f_n converges uniformly on all compact subsets of X.

   (b) Show that if $C(X,Y)$ carries the topology of pointwise convergence (i.e. the subspace topology defined via the obvious inclusion $C(X,Y) \subset Y^X$), then ev is not sequentially continuous in general.

   (c) Show that if $C(X,Y)$ carries the compact-open topology, then ev is always sequentially continuous.

   (d) ($*$) Show that if $C(X,Y)$ carries the compact-open topology and X is locally compact and Hausdorff, then ev is continuous.

(e) (∗) Show that every topology on $C(X, Y)$ for which ev is continuous contains the compact-open topology. (This proves that if X is locally compact and Hausdorff, the compact-open topology is the weakest topology for which the evaluation map is continuous.)
Hint: If $(f_0, x_0) \in \mathrm{ev}^{-1}(V)$ where $V \subset Y$ is open, then $(f_0, x_0) \in \mathcal{O} \times \mathcal{U} \subset \mathrm{ev}^{-1}(V)$ for some open $\mathcal{O} \subset C(X, Y)$ and $\mathcal{U} \subset X$. Is $\mathcal{U}_{K,V}$ a union of sets $\mathcal{O}$ that arise in this way?

(f) Show that for the compact-open topology on $C(\mathbb{Q}, \mathbb{R})$, $\mathrm{ev} : C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q} \to \mathbb{R}$ is not continuous.

8. (a) (∗) Given two pointed spaces (X, x) and (Y, y), prove that $\pi_1(X \times Y, (x, y))$ is isomorphic to the product group $\pi_1(X, x) \times \pi_1(Y, y)$.
Hint: Use the projections $p^X : X \times Y \to X$ and $p^Y : X \times Y \to Y$ to define a natural map from π_1 of the product to the product of π_1's, then prove that it is an isomorphism.

   (b) Generalize part (a) to the case of an infinite product of pointed spaces (with the product topology).

9. We can regard $\pi_1(X, p)$ as the set of base point preserving homotopy classes of maps $(S^1, \mathrm{pt}) \to (X, p)$. Let $[S^1, X]$ denote the set of homotopy classes of maps $S^1 \to X$, with no conditions on base points. (The elements of $[S^1, X]$ are called *free homotopy classes of loops* in X). There is a natural map

$$F : \pi_1(X, p) \to [S^1, X]$$

defined by ignoring base points. Prove:

   (a) F is surjective if X is path-connected.

   (b) (∗) $F([\alpha]) = F([\beta])$ if and only if $[\alpha]$ and $[\beta]$ are conjugate in $\pi_1(X, p)$.
Hint: If $H : [0, 1] \times S^1 \to X$ is a homotopy with $H(0, \cdot) = \alpha$ and $H(1, \cdot) = \beta$, and $t_0 \in S^1$ is the base point in S^1, then $\gamma := H(\cdot, t_0) : [0, 1] \to X$ begins and ends at p, and therefore also defines a loop. Compare α and the concatenation $\gamma \cdot \beta \cdot \gamma^{-1}$.

The conclusion is that if X is path-connected, F induces a bijection between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$. In particular, $\pi_1(X) \cong [S^1, X]$ whenever $\pi_1(X)$ is abelian.