PROBLEM SET 5
Due: 1.06.2023

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Note: The computation $\pi_1(S^1) \cong \mathbb{Z}$ was sketched in lecture last week, though some details of the proof were postponed and will not be covered in earnest for another few weeks. Nonetheless, you are free to treat $\pi_1(S^1) \cong \mathbb{Z}$ as a known fact for the foreseeable future, and it will frequently be needed in exercises.

Problems

1. For a point $z \in \mathbb{C}$ and a continuous map $\gamma : [0,1] \to \mathbb{C}\backslash\{z\}$ with $\gamma(0) = \gamma(1)$, one defines the *winding number* of γ about z as
$$\mathrm{wind}(\gamma; z) = \theta(1) - \theta(0) \in \mathbb{Z}$$
where $\theta : [0,1] \to \mathbb{R}$ is any choice of continuous function such that
$$\gamma(t) = z + r(t)e^{2\pi i\theta(t)}$$
for some function $r : [0,1] \to (0,\infty)$. Notice that since $\gamma(t) \neq z$ for all t, the function $r(t)$ is uniquely determined, and requiring $\theta(t)$ to be continuous makes it unique up to the addition of a constant integer, hence $\theta(1) - \theta(0)$ depends only on the path γ on not on any additional choices. One of the fundamental facts about winding numbers is their important role in the computation of $\pi_1(S^1)$: as we saw in lecture, viewing S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, the map
$$\pi_1(S^1, 1) \to \mathbb{Z} : [\gamma] \mapsto \mathrm{wind}(\gamma; 0)$$
is an isomorphism to the abelian group $(\mathbb{Z}, +)$. Assume in the following that $\Omega \subset \mathbb{C}$ is an open set and $f : \Omega \to \mathbb{C}$ is a continuous function.

   (a) Suppose $f(z) = w$ and $w \notin f(\mathcal{U}\backslash\{z\})$ for some neighborhood $\mathcal{U} \subset \Omega$ of z. This implies that the loop $f \circ \gamma_\epsilon$ for $\gamma_\epsilon : [0,1] \to \Omega : t \mapsto z + \epsilon e^{2\pi i t}$ has image in $\mathbb{C}\backslash\{w\}$ for all $\epsilon > 0$ sufficiently small, hence $\mathrm{wind}(f \circ \gamma_\epsilon; w)$ is well defined. Show that for some $\epsilon_0 > 0$, $\mathrm{wind}(f \circ \gamma_\epsilon; w)$ does not depend on ϵ as long as $0 < \epsilon \leqslant \epsilon_0$.

   (b) $(*)$ Show that if the ball $B_r(z_0)$ of radius $r > 0$ about $z_0 \in \Omega$ has its closure contained in Ω, and the loop $\gamma(t) = z_0 + re^{2\pi i t}$ satisfies $\mathrm{wind}(f \circ \gamma; w) \neq 0$ for some $w \in \mathbb{C}$, then the equation $f(z) = w$ has a solution $z \in \Omega$ with $|z - z_0| < r$.
   Hint: Recall that if we regard elements of $\pi_1(X,p)$ as pointed homotopy classes of maps $S^1 \to X$, then such a map represents the identity in $\pi_1(X,p)$ if and only if it admits a continuous extension to a map $\mathbb{D}^2 \to X$. Define X in the present case to be $\mathbb{C}\backslash\{w\}$.

   (c) Prove the Fundamental Theorem of Algebra: every nonconstant complex polynomial has a root.
   Hint: Consider loops $\gamma(t) = Re^{2\pi i t}$ with $R > 0$ large.

   (d) $(*)$ We call $z_0 \in \Omega$ a *zero* of $f : \Omega \to \mathbb{C}$ if $f(z_0) = 0$, and z_0 is called an *isolated zero* of f if additionally $0 \notin f(\mathcal{U}\backslash\{z_0\})$ for some neighborhood $\mathcal{U} \subset \Omega$ of z_0. Let us say that an isolated zero has *order* $k \in \mathbb{Z}$ if $\mathrm{wind}(f \circ \gamma_\epsilon; 0) = k$ for $\gamma_\epsilon(t) = z_0 + \epsilon e^{2\pi i t}$ and $\epsilon > 0$ small (recall from part (a) that this does not depend on the choice of ϵ if it is small enough). Show that if $k \neq 0$, then for any neighborhood $\mathcal{U} \subset \Omega$ of z_0, there exists $\delta > 0$ such that every continuous function $g : \Omega \to \mathbb{C}$ satisfying $|f - g| < \delta$ everywhere has a zero somewhere in $\mathcal{U}$.

(e) ($*$) Find an explicit example in which $f : \Omega \to \mathbb{C}$ has an isolated zero of order 0 at some point $z_0 \in \Omega$ and f admits arbitrarily close perturbations g (meaning functions that satisfy $|f - g| < \epsilon$ for $\epsilon > 0$ small) that have no zeroes at all.
Hint: Write f as a continuous function of x and y where $x + iy \in \Omega$. You will not be able to find an example for which f is holomorphic—they do not exist!

General advice: Throughout this problem, it is important to remember that $\mathbb{C}\backslash\{w\}$ is homotopy equivalent to S^1 for every $w \in \mathbb{C}$. Thus all questions about $\pi_1(\mathbb{C}\backslash\{w\})$ can be reduced to questions about $\pi_1(S^1)$.

2. For each of the following spaces X and subspaces $A \subset X$, determine whether A is a retract or a deformation retract of X, or neither. Justify your answer in each case by either describing a (deformation) retraction[1] or saying something about fundamental groups.

   (a) ($*$) $A = S^1 \times \{\text{pt}\}$ in $X = S^1 \times S^1$

   (b) $A = \{(x_0, 0)\}$ in $X = \big([0,1] \times \{0\}\big) \cup \big(\{0\} \times [0,1]\big) \cup \big(\bigcup_{n\in\mathbb{N}}\{2^{-n}\} \times [0,1]\big)$, where $0 < x_0 < 1$

   (c) ($*$) $A = \big(S^1 \times \{y\}\big) \cup \big(\{x\} \times S^1\big)$ in $X = \big(S^1 \times S^1\big) \backslash\{(x_0, y_0)\}$ with $x_0 \neq x$ and $y_0 \neq y$

3. Here is a useful fact from linear algebra known as *polar decomposition*: every invertible real matrix $\mathbf{A} \in \mathrm{GL}(n, \mathbb{R})$ can be written as $\mathbf{PR}$, where $\mathbf{R}$ is orthogonal and $\mathbf{P}$ is symmetric positive-definite. To see this, notice that $\mathbf{AA}^T$ is always symmetric and positive-definite, thus it can be written as $\mathbf{M\Lambda M}^T$ for some orthogonal $\mathbf{M}$ and diagonal $\mathbf{\Lambda}$ with positive entries, making it possible to define powers $(\mathbf{AA}^T)^p = \mathbf{M\Lambda}^p\mathbf{M}^T$ for every $p \in \mathbb{R}$. Then defining $\mathbf{P} := (\mathbf{AA}^T)^{1/2}$, it is not hard to verify that $\mathbf{R} := \mathbf{P}^{-1}\mathbf{A}$ is orthogonal.

   (a) Use polar decomposition to show that the group $\{\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) \mid \det(\mathbf{A}) > 0\}$ admits a deformation retraction to the special orthogonal group $\mathrm{SO}(n)$ for every $n \in \mathbb{N}$.[2]

   (b) Identifying S^1 with the quotient group $\mathbb{R}/\mathbb{Z}$, show that every loop $\mathbf{A} : S^1 \to \mathrm{GL}(2, \mathbb{R})$ passing through the identity matrix is homotopic in $\mathrm{GL}(2, \mathbb{R})$ to a loop of rotations

$$\mathbf{A}(t) = \begin{pmatrix} \cos(2\pi kt) & -\sin(2\pi kt) \\ \sin(2\pi kt) & \cos(2\pi kt) \end{pmatrix}$$

   for some $k \in \mathbb{Z}$, and k is uniquely determined by $\mathbf{A} : S^1 \to \mathrm{GL}(2, \mathbb{R})$.
   Hint: What is $\mathrm{SO}(2)$ *homeomorphic to?*

[1] In most cases, a good picture can suffice as a description—there is no need to write down maps or homotopies in precise formulas.

[2] Here we assume $\mathrm{GL}(n, \mathbb{R})$ carries its natural topology as an open subset of the space of all real n-by-n matrices (a vector space isomorphic to $\mathbb{R}^{n^2}$).