PROBLEM SET 7
Due: 15.06.2023

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

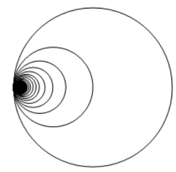
Note: You may continue to treat the computation $\pi_1(S^1) \cong \mathbb{Z}$ *as a known fact, even though we have not proved it in full detail yet.*

Problems

1. It is commonly said that whenever X and Y are both path-connected and are otherwise "reasonable" spaces, their wedge sum $X \vee Y$ satisfies $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$ We've seen for instance that this is true when X and Y are both circles. The goal of this problem is to understand slightly better what "reasonable" means in this context, and why such a condition is needed.

   (a) Call a pointed space (X, x) *nice*[1] if x has an open neighborhood that admits a deformation retraction to x. Show that for any collection $\{(X_\alpha, x_\alpha)\}_{\alpha \in J}$ of nice pointed spaces, $\pi_1(\bigvee_{\alpha \in J} X_\alpha) \cong \mathop{\ast}_{\alpha \in J} \pi_1(X_\alpha, x_\alpha)$, where we use the natural choice of base point on $\bigvee_{\alpha \in J} X_\alpha$.

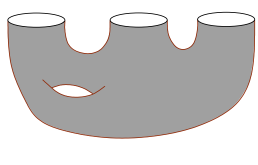
   (b) Here is an example of a space that is not "nice" in the sense of part (a): the so-called *Hawaiian earring* $H \subset \mathbb{R}^2$ can be defined as the subset of $\mathbb{R}^2$ consisting of the union for all $n \in \mathbb{N}$ of the circles of radius $1/n$ centered at $(1/n, 0)$. As usual, we assign to this set the subspace topology induced by the standard topology of $\mathbb{R}^2$. It is tempting to liken H to the wedge sum $X := \bigvee_{n=1}^{\infty} S^1$, since both are unions of countably infinite collections of circles that all intersect each other at one point. Show however that X is nice, and H is not: in particular, the point $(0, 0)$ does not have any simply connected open neighborhood in H.



   (c) For the spaces H and X in part (b), show that there exists a *surjective* continuous map $S^1 \to H$, but continuous maps $S^1 \to X$ are never surjective.
   Hint: In H*, start at* $(0, 0)$ *and traverse the largest circle first, then continue to smaller circles.*

   Comment: Part (c) gives a hint that $\pi_1(H)$ *is a strictly larger group than* $\pi_1(X) \cong \mathop{\ast}_{n=1}^{\infty} \mathbb{Z}$*. If you're curious for more details, see page 49 of Hatcher and the reference mentioned there.*

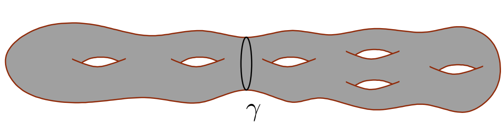
2. For integers $g, m \geqslant 0$, let $\Sigma_{g,m}$ denote the compact surface obtained by cutting m disjoint disk-shaped holes out of the closed orientable surface with genus g. (By this convention, $\Sigma_g = \Sigma_{g,0}$.) The boundary $\partial\Sigma_{g,m}$ is then a disjoint union of m circles, e.g. the case with $g = 1$ and $m = 3$ might look like the picture at the right.



   (a) $(*)$ Show that $\pi_1(\Sigma_{g,1})$ is a free group with $2g$ generators, and if $g \geqslant 1$, then any simple closed curve parametrizing $\partial\Sigma_{g,1}$ represents a nontrivial element of $\pi_1(\Sigma_{g,1})$.[2]
   Hint: Think of Σ_g *as a polygon with some of its edges identified. If you cut a hole in the middle of the polygon, what remains admits a deformation retraction to the edges. Prove it with a picture.*

[1] Not a standardized term, I made it up.
[2] Terminology: one says in this case that $\partial\Sigma_{g,1}$ is *homotopically nontrivial* or *essential*, or equivalently, *not nullhomotopic.*

(b) ($*$) Assume γ is a simple closed curve separating Σ_g into two pieces homeomorphic to $\Sigma_{h,1}$ and $\Sigma_{k,1}$ for some $h, k \geqslant 0$. (The picture at the right shows an example with $h = 2$ and $k = 4$.) Show that the image of $[\gamma] \in \pi_1(\Sigma_g)$ under the natural projection to the abelianization of $\pi_1(\Sigma_g)$ is trivial.



Hint: What does γ look like in the polygonal picture from part (a)? What is it homotopic to?

(c) ($*$) In the following, we abbreviate $[x, y] := xyx^{-1}y^{-1}$ and $\prod_{i=1}^{N} x_i := x_1 x_2 \ldots x_N$, thus

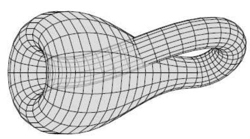
$$G_g := \left\{ a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = e \right\}$$

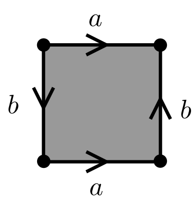
is the standard presentation of $\pi_1(\Sigma_g)$. Show that if $J \subset \{1, \ldots, g\}$ is a nonempty proper subset, then there exists a homomorphism from G_g to a free group on two generators that sends the element $\prod_{i \in J} [a_i, b_i] \in G_g$ to something nontrivial. Deduce that G_g is not abelian for every $g \geqslant 2$.
Hint: Once you've specified what your homomorphism does on each of the generators a_i, b_i, this determines a homomorphism defined on the free group $F_{\{a_1, b_1, \ldots, a_g, b_g\}}$, but you need to make sure it is trivial on the normal subgroup generated by the relation. Make your definition as simple as possible.

(d) ($*$) Show that for $h, k > 0$, the curve γ in part (b) represents a nontrivial element of $\pi_1(\Sigma_g)$.

(e) Generalize part (a): show that if $m \geqslant 1$, $\pi_1(\Sigma_{g,m})$ is a free group with $2g + m - 1$ generators.

3. The first of the two pictures at the right shows one of the standard ways of representing the *Klein bottle*[3] as an "immersed" (i.e. smooth but with self-intersections) surface in $\mathbb{R}^3$. As a topological space, the technical definition is





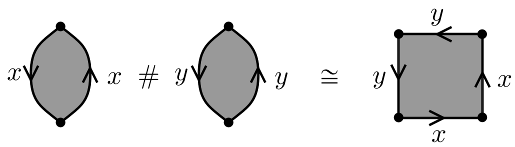
$$\mathbb{K}^2 = [0, 1]^2 / \sim$$

where $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, 1 - t)$ for every $s, t \in [0, 1]$. This is represented by the square with pairs of sides identified in the rightmost picture; notice the reversal of arrows, which is why $\mathbb{K}^2 \neq \mathbb{T}^2$! A theorem we proved in lecture thus implies that $\pi_1(\mathbb{K}^2)$ is isomorphic to the group

$$G := \{ a, b \mid aba^{-1}b = e \}.$$

(a) ($*$) Consider the subset $\ell = \{[(s, t)] \in \mathbb{K}^2 \mid t = 1/4 \text{ or } t = 3/4\}$ in $\mathbb{K}^2$. Show that ℓ is a simple closed curve which separates $\mathbb{K}^2$ into two pieces, each homeomorphic to the Möbius band $\mathbb{M}^2 := \{(e^{i\theta}, \tau e^{i\theta/2}) \in S^1 \times \mathbb{C} \mid \theta \in [0, 2\pi],\ \tau \in [-1, 1]\}$. Use this decomposition to show via the Seifert-van Kampen theorem that $\pi_1(\mathbb{K}^2)$ is also isomorphic to $G' := \{ c, d \mid c^2 = d^2 \}$.

(b) Recall that $\mathbb{RP}^2$ can be constructed by gluing $\mathbb{M}^2$ to a disk $\mathbb{D}^2$, so conversely, $\mathbb{RP}^2 \backslash \mathring{\mathbb{D}}^2 \cong \mathbb{M}^2$. Part (a) implies therefore that $\mathbb{K}^2$ is homoemorphic to the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$. Now, viewing $\mathbb{RP}^2$ as a polygon with two (curved) edges that are identified, imitate the argument we carried out for Σ_g in lecture to derive a different presentation for $\mathbb{K}^2$ as shown in the figure below, and deduce that $\pi_1(\mathbb{K}^2)$ is also isomorphic to $G'' := \{ x, y \mid x^2 y^2 = e \}$.



(c) For the groups G, G' and G'' above, find explicit isomorphisms of their abelianizations to $\mathbb{Z} \oplus \mathbb{Z}_2$. Then find explicit isomorphisms from each of G, G' and G'' to the others.

[3] If you think my glass Klein bottle is cool, you can buy your own at `http://www.kleinbottle.com/`.