Symplectic field theory
Problem set 1

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To be discussed on the 23rd of April

This problem set should cover the first two weeks. The first exercise session is on the 16th of April but it will look more like a lecture to complement the content of Prof. Wendl's lectures. Consequently, there are no problems assigned for the 16th.

These exercises, to be discussed on the 23rd, should familiarize you with symplectic and contact manifolds and explore some topics mentioned on the first week. The main goal of this set is to motivate some of the questions that we will answer throughout the course (and also in the symplectic seminar). I would say that it is more important to give some thought to most problems and convince yourself of the answers rather than producing a detailed answer for a few.

Problem 1. Some examples and non-examples of symplectic manifolds. Convince yourself the following are symplectic manifolds:

a. Cotangent bundles T^*M of smooth manifolds M,

b. complex projective space $\mathbb{CP}^n$,

c. projective varieties (i.e. things cut out by homogeneous polynomials in $\mathbb{CP}^n$),

d. product of symplectic manifolds,

e. all orientable surfaces (in particular $\mathbb{S}^2$ and $\mathbb{T}^2$).

Show that the following manifolds do not carry any symplectic form:

f. The $2n$-sphere for $n > 1$.

g. Complex projective plane with orientation reversed $\overline{\mathbb{CP}}^2$.

h. The self-connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$.[1]

Remark. Determining which closed manifolds admit symplectic structures (and the geometry of the space of such structures when non-empty) is a very hard problem. On the other hand, however, it is well understood for open manifolds, where an h-principle holds. The theory of J-holomorphic curves in 4-dimensions is the most effective tool known to attack this problem. In a few months, using ideas from the course, we could show that the symplectic forms on $S^2 \times S^2$ are parametrized up to symplectomorphism by the symplectic areas of the factor spheres $S^2 \times pt$ and $pt \times S^2$, i.e. by $(0, \infty) \times (0, \infty)$. Much of what is known in these questions of existence and uniqueness of symplectic forms is due to is Seiberg-Witten-Taubes theory (in dimension 4).

[1] Hint: use the Hirzebruch signature theorem (black box): any almost complex 4-manifolds (M, J) must satisfy the identity $c_1(M, J)^2 = 2\chi(M) + 3\sigma$, where σ is the signature of the intersection form.

Problem 2. Some examples of contact manifolds.[2]

a. Consider $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ and corresponding polar coordinates $(r_1, \theta_1, \ldots, r_n, \theta_n, z)$. Show that

$$\alpha_{\text{std}} = dz + \Sigma_{i=1}^{n} x_i dy_i - y_i dx_i = dz + \sum_{i=1}^{n} r_i^2 d\theta_i$$

is a contact form. Visualize $\xi_{\text{std}} = \ker \alpha_{\text{std}}$ for $n = 1$. Show that the form $\alpha_{\text{std}} = \sum_{i=1}^{2n} x_i dy_i - y_i dx_i$ is contact on the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{R}^{2n+2}$. With these forms, convince yourself that $\mathbb{S}^{2n+1} \setminus \infty$ is contactomorphic to $\mathbb{R}^{2n+1}$.

b. With the coordinates above, consider the form $\alpha_{\text{OT}} = \cos r dz + r \sin r d\theta$ on $\mathbb{R}^3$, show that it is contact and visualize $\xi_{\text{OT}} = \ker \alpha_{\text{OT}}$.

c. Recall that the Lens spaces $L(a, b)$, for a and b coprime integers, are defined as $\mathbb{S}^3 \subseteq \mathbb{C}^2_{z_1, z_2}$ modulo the $\mathbb{Z}/a$-action given by $(z_1, z_2) \mapsto (e^{2\pi i/a} z_1, e^{2\pi b i/a} z_2)$. Show that these are contact manifolds. *Bonus:* assuming we already know $(\mathbb{S}^3 . \xi_{\text{std}})$ is tight, show that there is a tight contact form on every $L(a, b)$.

d. Similarly, show that $\mathbb{RP}^{2n-1}$ is contact.

e. Star-shaped hypersurfaces in $\mathbb{C}^n$ are hypersurfaces on which the radial vector field is positively transverse. Show that these are contact manifolds.

f. Show that unit cotangent bundles $\mathbb{S}^* M$ of smooth manifolds M (i.e. choose a Riemannian metric on M, which gives a metric on $T^* M$, and consider the loci of unit vectors over M, this is a sphere bundle) are contact manifolds.

g. We have shown that $\mathbb{T}^{2n+1} = \mathbb{S}^* \mathbb{T}^{2n+1}$ is contact. Let's write down more contact forms on the 3-torus. Consider $\mathbb{T}^3_{(\rho, \phi, \theta)}$ and contact forms $\alpha_k = \cos(2\pi k \rho) d\theta + \sin(2\pi k \rho) \phi$ for each integer $k \geq 1$. Show that each α_k is a contact form and visualize the contact distribution $\xi_k = \ker \alpha_k$ (suggestion: consider $[0, k] \times \mathbb{T}^2$). Show that all ξ_k are actually homotopic as plane-fields.

Remark. By the end of the course, and a little bit of extra theory, we will be able to use holomorphic curves to show that the standard contact 3-sphere is tight or that overtwisted contact structures on 3-manifolds and the standard tight 3-sphere have (contractible) periodic Reeb orbits. What we will explicitly show in the lectures, using SFT, is that the contact structures ξ_k on the 3-torus are not contactomorphic, in spite of having just shown that their "formal invariants" are the same. This is also an example of how tight contact structures do not obey the h-principle. Moreover, these class of contact structures pave the ground for establishing a "hierarchy of contact manifolds".

Problem 3. Dynamics of contact manifolds.

a. Ellipsoids. Write down the contact form and Reeb vector field for the ellipsoids $E(a, b)$:

$$E(a, b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi}{a}|z_1|^2 + \frac{\pi}{b}|z_2|^2 \leq 1\}.$$

[2] A good tool for exploring some of these examples is the following: let M be a closed oriented hypersurface of a symplectic manifold (W, ω). A Liouville vector field in W is a vector field V that preserves the symplectic form (i.e. $\mathfrak{L}_V \omega = \omega$). We say that M is of contact-type if there is a Liouville vector field on W (or just near M, in fact) positively transverse to M. Taking the dual of V on M, i.e. $\iota_V \omega|_{TM}$, produces a contact form on M, so contact-type implies contact. The converse is also true by the symplectization construction covered in the lecture.

Show that the Reeb flow on the unit sphere $E(1,1)$ parametrizes the Hopf fibration (sort of a $\mathbb{P}^1$-family of orbits) and that if a/b is irrational there are only two closed Reeb orbits. *Bonus:* Does this follow through to Lens spaces?[3]

b. Riemannian geometry as contact geometry. Let (M,g) be a Riemannian manifold and let G be the geodesic vector field on TM, that is $G : TM \longrightarrow TTM$ defined by $G(x,v) = (v,0)$. Implicit here is the splitting $TTM = VM \oplus \pi^*TM$ given by the Levi-Civita connection.

   i. Show that closed geodesics lift to unique closed orbits of the geodesic flow on the unit tangent bundle.

   ii. Show that under the isomorphism $\mathbb{S}M \to \mathbb{S}^*M$ induced by the metric g, the vector field G and R get identified.

   iii. Produce examples of contact manifolds with no contractible Reeb orbits as well as one with finitely many.

Remark. All star-shaped hypersurfaces in $\mathbb{C}^2$ have the same contact structure, that of the standard sphere. Ellipsoids are a particularly symmetric example and we already see that the dynamics of different contact forms on the standard contact 3-sphere can change a lot. We will revisit these examples in the future. The SFT of the unit cotangent of hyperbolic manifolds provides very strong restrictions to a problem that is otherwise very hard: when does a given manifold embed as a Lagrangian in a given symplectic manifold? In the problem sets will show that a large class of symplectic manifolds do not have hyperbolic Lagrangians.

Problem 4. Some examples of symplectic cobordisms between contact manifolds.

a. Show that the 3-sphere $\mathbb{S}^3$ and the 3-torus $\mathbb{T}^3$ are fillable (i.e. admit a cobordism from the empty set).

b. Show that if a contact 3-manifold M is fillable, there is a symplectic cobordism from $\mathbb{S}^3$ to M.

c. From the above, there is a symplectic cobordism from $\mathbb{S}^3$ to $\mathbb{T}^3$, find one from $\mathbb{T}^3$ to $\mathbb{S}^3$. This shows that the symplectic cobordism relation is not antisymmetric (so it is not a partial order).

d. Show that $\mathbb{RP}^{2n+1}$ is fillable but that it cannot be filled by something that has the type of an $(n+1)$-dimensional CW-complex (so it is not Weinstein fillable) for $n > 1$.[4]

Remark. In general, that there is a symplectic cobordism from (M,ξ_-) to (M,ξ_+) does not mean that there is one from (M,ξ_+) to (M,ξ_-). With tools introduced later on in the semester we will be able to do the following exercise: there is a symplectic cobordism from (S^3,ξ_{ot}) to (S^3,ξ_{std}) but none the other way around. There are deep reasons for this, to start motivating it, we present a central question: does the existence of a symplectic cobordism between contact manifolds (eg. a filling) impose contact geometric/topological restrictions? More generally, does it "transfer" information? An example, which will be available after the course (or during the seminar), is that fillable contact manifolds are tight. SFT was basically created to understand which rigid properties are encoded in cobordisms.

Problem 5. A generalization of contact-type hypersurface (in which SFT can be set-up). Let (W,ω) be a $2n$-dimensional symplectic manifold and $M \subseteq W$ a closed oriented hypersurface with $\omega_M := \omega|_{TM} \in \Omega^2(M)$ of maximal rank.

[3]Now that there are two or infinitely many. It is a difficult theorem that the presence of exactly two closed orbits on a contact 3-manifold forces the the 3-manifold to be spherical. Even a deeper theorem claims that there are either two or infinitely many for any given contact structure (with a mild topological assumption). The course will indeed provide a head-start in understanding these very recent deep results.

[4]If W fills $M = \mathbb{RP}^{2n+1}$ and W had the homotopy type of an $(n+1)$-dimensional CW-complex, show that $H^2(W,M;Z/2) = 0$ and that there is a class $a \in H^1(W;Z/2)$ such that in $\partial a \in H^1(M;Z/2)$ generates it. Then, $(\partial a)^{2n+1}$ is not zero but $a^{2n+1} \in H^{2n+1}(W;Z/2)$ is also not zero, which is a contradiction.

a. Observe that $\ell_\omega = \ker \omega_M \subseteq TM$ is an oriented line field. Show that a $\lambda \in \Omega^1(M)$ that is positive on ℓ_ω is equivalent to a $\lambda \in \Omega^1(M)$ such that $\lambda \wedge \omega^{n-1} > 0$ everywhere on M.

b. Given $\lambda \in \Omega^1(M)$, we can use the symplectic form to define a vector field V of TW over M. Conversely, given such a vector field V we can define a 1-form on M $\lambda = \iota_V \omega$. Show that $\iota_V \omega|_{TM}$ is positive on ℓ_ω if and only if V is positively transverse to M.

c. Show that the set of such 1-forms (as in *a.*) is convex and also non-empty (using partitions of unity).

d. Given a 1-form as in *a.*, show that M admits a symplectic tubular neighbourhood $(-\varepsilon, \varepsilon)_r \times M$ in which $\omega = \omega_M + d(r\lambda)$. To do this use the vector field dual to λ to find the collar and then use Moser's trick to make ω and $\omega_M + d(r\lambda)$ match.

Remark. The content of this problem will become useful later when we set up SFT for symplectic cobordisms between a certain class of odd-dimensional manifolds that is broader that the contact ones. This will, for example, make Hamiltonian Floer theory a particular case of the set the set-up in the lecutres. The way Moser's trick is used in this exercise is bread and butter for symplectic geometers. *Bonus:* prove the Morse lemma using Moser's trick.

Problem 6. A cute application of Moser's trick. Let (M, vol) be a closed smooth manifold with a volume form vol, D its group of (orientation preserving) diffeomorphisms and V its group of (orientation preserving) volume preserving diffeomorphisms.

a. Show that if vol' is another volume form on M with the same volume (i.e. such that $\int_M vol = \int_M vol'$), then there is a diffeomorphism $f : M \to M$ such that $f^* vol' = vol$. Hint: Use Moser's trick.

b. Show that the group D is homotopy equivalent to V. Hint: let D act on the space of volume forms of the same volume and think of homogeneous spaces.

c. Consequently, for closed symplectic 2-manifolds the symplectomorphism groups are computable from the diffeomorphism groups. Do you know how to compute any of those?

Remark. It is a hard problem to compute symmetry groups with respect to geometric/topological structures. For example, the compactly supported diffeomorphism group of $\mathbb{R}^4$ is unknown, while the compactly supported group of symplectomorphisms of $\mathbb{R}^4$ is known to be contractible, but it is a non-trivial theorem requiring tools like the ones we will develop. The general line of thought behind this exercise underlies one of the two basic strategies in most of the known computations of groups of symplectic symmetries of symplectic manifolds.