Symplectic field theory
Problem set 3

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To be discussed on the 7th of May

This is a differential topology and linear algebra work sheet. The first and third problems highlight some interesting features or deep facts of holomorphic curve theory and symplectic geometry, while the second problem helps digest the spectral flow theory.

In problem 1, we start with some classical differential topology that will covertly serve as a blueprint of our strategy to show moduli spaces are manifolds (generically). Our goal is to show that transverse intersections are smooth and that transversality is a generic property (along with a couple of consequences). The second problem is meant to help understand some of the geometry behind the spectral flow in the finite-dimensional setting. Lastly, the third problem studies the group of linear symplectic maps, which appears in a few different dynamical and geometric guises in the lecture (e.g. Darboux's theorem, linearized Reeb flow and contractibility of the space of almost complex structures).

Problem 1. Transversality in finite dimensions. In this problem X, Y and Z will be manifolds (without boundary) and U_n an open subset of $\mathbb{R}^n$ containing 0. If Y and Z are submanifolds of X, we say that they **intersect transversely** (and write $Y \pitchfork Z$) if for every $x \in Y \cap Z$ we have $T_xY + T_xZ = T_xX$. Note that transverse submanifolds such that $\dim Y + \dim Z < \dim X$ have empty intersection. Similarly, a smooth map $f : Y \longrightarrow X$ is **transverse to** Z if $f_*T_yY + T_xZ = T_xX$ for every $y \in Y$ such that $x = f(y) \in Z$.

a. Show that if $f : Y \longrightarrow X$ is a smooth map and $Z \subseteq X$ a submanifold such that f is transverse to Z, then

$$f \cap Z := \{y \in Y \mid f(y) \in Z\} \subseteq Y$$

is a submanifold of codimension

$$\operatorname{codim}_Y f \cap Z = \operatorname{codim}_X Z.$$

It may help to first do the case when f is an embedding so Y is a submanifold of X transverse to Z and $Y \pitchfork Z$ is then a submanifold of X of $\operatorname{codim} Y \cap Z = \operatorname{codim} Y + \operatorname{codim} Z$.

The main ingredient in showing that transversality is a generic property is Sard's theorem. Let $f : Y \to X$ be a smooth map, a critical point of f is a point $y \in Y$ such that T_yf is not surjective and a critical value if the image in X under f of a critical value. Otherwise, points in Y or values in X that are not critical are called regular. Sard's theorem says almost all values are regular values.

Sard's theorem. *The set of critical values of a smooth map has measure 0.*[1]

[1]Funnily enough, we do not need a measure to define a set of measure 0: a set $A \subseteq X$ is said to be of measure 0 if in all charts of X has measure 0. Closed sets A of measure zero are nowehere dense. In fact, it suffices to assume that A is the countable union of compact sets, which are nowhere dense and then so is A by the Baire category theorem. Later on in the course we will use the word meager to generalize the notion of measure zero. The complement of a meager set is comeager and it is defined as a set that contains the countable intersection of open and dense sets. Countable intersections of comeager sets are comeager and by the Baire category theorem comeager sets are dense.

Take a second to convince yourself that this is intuitive (the differential of a map not being surjective is the same as not being full-rank, which imposes a constraint of at least one dimension).

b. Show this easier case of Sard's theorem: if $f : Y \longrightarrow X$ is a smooth map and $\dim Y < \dim X$, then its set of critical values has measure zero.

   Hint: You can take as a black box the measure theoretic fact that a $\mathcal{C}^1$ map $f : \mathbb{R}^k \to \mathbb{R}^n$ for $k \leq n$ sends sets of measure zero to sets of measure zero.

To really talk about "generic" transversality of smooth maps, one should define a topology on $\mathcal{C}^\infty(Y, X)$. This is given by $\mathcal{C}^\infty_{\text{loc}}$-convergence (uniform convergence on compact sets of all derivatives). Similarly, to talk about families of smooth maps one should say what a smooth map $U_n \longrightarrow \mathcal{C}^\infty(Y, X)$ is. This is *defined* as a smooth map $F : Y \times U_n \longrightarrow X$ and we say this is a smooth family $F = (f_s)_{s \in U_n}$ or that f_s are a smooth (n-parameter) deformation of f_0.

c. * Consider the following smooth family $F = (f_s : Y \longrightarrow X)_{s \in U_n}$ of smooth maps and $Z \subseteq X$ a submanifold of X. Consider the intersection $F \cap Z \subseteq Y \times U_n$ and the projection $\pi : F \cap Z \longrightarrow U$ that projects $(y, s) \longmapsto s$. Show that if F is transverse to Z, then f_{s_0} is transverse to Z if and only if s_0 is a regular value of π. Deduce from this that f_s is transverse to Z for almost all s.

   Hint: That regular values of π give parameters at which the smooth map is transverse is basically an algebraic manipulation. The fastest and most conceptual way to do it is using the snake lemma.

d. Given a submanifold $Z \subseteq \mathbb{R}^n$ and a smooth map $f : Y \longrightarrow \mathbb{R}^n$, for any $\varepsilon > 0$ there exists a smooth map $f_\varepsilon : Y \longrightarrow \mathbb{R}^n$ that is transverse to Z and $|f(y) - f_\varepsilon(y)| < \varepsilon$ for all $y \in Y$.

   Philosophical hint: Be brute, try to make a big family F by adding as many parameters as needed.

Using the Whitney embedding theorem (or easier versions of it) one can see replace $\mathbb{R}^n$ with a manifold X in the statement above. This is what we take to mean "generic transversality".[2] Here are two applications of transversality in differential topology.

e. * Convince yourself that the complement of any submanifold of X of codimension at least 2 is connected.

f. * Show that complex vector bundles over closed surfaces (2-manifolds) decompose into a sum of complex line bundles (real rank 2) and that the number of zeros of a generic section of a line bundle is finite.[3]

g. Bonus: We have shown transversality is generic, it is also stable (i.e. a sufficiently small perturbation of a transverse map will continue to be transverse; the properties of being an embedding or an immersion are also stable). Do you see which point of our argument actually proves stability as well?

Remark. For the first time in the lectures we were exposed to a non-trivial use of transversality: a path will generically avoid submanifolds of codimension 2 or larger. While there we were discussing Banach manifolds of finite codimension, here we have focused on the finite-dimensional case first.[4] The usefulness of transversality has been present in geometry since the beginning: a classical way to say that two submanifolds intersect transversely is to say they are in general position.[5] In classical differential topology it's still very useful, as we can see from *f.*, for example.

[2]One can do better: given $Z \subseteq X$ a submanifold of X, the set of maps in $\mathcal{C}^\infty(Y, X)$ transverse to Z is comeager. However, from what we have shown, this claim closer to point-set topology than it is to differential topology.

[3]Bonus: counting these zeros with multiplicities produces a count invariant under homotopies of sections. This is the blueprint to define the first Chern number.

[4]It is very likely that a future problem focuses on how to generalize this to the Fredholm/Banach setting.

[5]For example: any two lines in general position (in particular: generically) in $\mathbb{R}^2$ intersect at a point but in $\mathbb{R}^3$ are disjoint. This also implies that three generic points in $\mathbb{R}^2$ are form a triangle.

In our case, we will make heavy use of it (in the Fredholm setting) to show that the solution space to the Cauchy-Riemann equation is well-behaved. The way I have written *c.* and *d.* is meant to mimic the infinite-dimensional argument we will give.

Remark'. This comment deserves its own remark. Given $f : Y \longrightarrow X$ and $g : Z \longrightarrow X$ smooth maps, we can make sense of $f \pitchfork g$ as we did before. In that case,

$$f \cap g = \{(y, z) \in Y \times Z \mid f(y) = g(z)\}$$

is a smooth manifold that embedds in M via both f and g of codimension $2n-(k+\ell)$. This follows easily from *a.* and rephrasing the problem as $f \times g$ intersecting the diagonal in $X \times X$. Consider now the category of smooth manifolds (objects are smooth maps), the set defined as $f \cap g$ realizes the category theoretic fiber product $Y{}_f\times_g Z$ and, when $f \pitchfork g$, the fiber product is again a smooth manifold. In general, however, this category does not contain fiber-products. Similarly, it does not contain quotients in general. One can enlarge it in clever ways so that it does, but then one must deal with derived smooth stacks. May seem gnarly, indeed, but it's the idea behind some major progress in the last 10 years in the field. Later on we will have an exercise elaborating on this a little bit.

Problem 2. Spectral flow in finite-dimensions. Consider the space $E = \mathrm{End}(\mathbb{R}^n)$ of linear maps $\mathbb{R} \to \mathbb{R}$ and $E^s = \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^n) \subseteq \mathrm{End}(\mathbb{R}^n)$ the symmetric linear transformations. Denote by E_k and $E^s_k = E^s \cap E_k$ the subset of matrices of a rank $0 \leq k \leq n$ and E_* and E^s_* the corresponding sets of non-invertible matrices.

a. Following the lectures, show that E_k is a submanifold of E of codimension $(n-k)^2$. Adapt the arguments for the symmetric case: $E^s_k \subseteq E$ is a smooth submanifold of codimension $\frac{(n-k)(n-k+1)}{2}$.

b. * Show that E^s_{n-1} admits a canonical co-orientation. Why doesn't the same argument work for E_{n-1}?

c. Consider a path $A : (-\varepsilon, \varepsilon) \to E$ is smooth such that $A(0) \in E_{n-1}$ and $A(t)$ is invertible for $t \neq 0$. Show that the intersection with E_{n-1} is transverse if and only if $\frac{d}{dt} \det A(t)|_{t=0} \neq 0$. The same works for symmetric case, with more information: since E^s_{n-1} is co-oriented, each intersection either counts positively or negatively, show that this is given by $\frac{d}{dt} \det A(t)|_{t=0}$ being positive or negative respectively.[6]

d. Visualize E_0 and E_1 in E, as well as E^s_0 and E^s_1 in E^s, when $n = 2$. Show that E_1 is connected and E^s_1 is not. Show that E^s_1 is not connected for $n \geq 2$.

We now set up the count that gives $\mathbb{Z}/2$-version of the spectral flow for matrices and the spectral flow for symmetric matrices, which is the finite-dimensional version of the one done in class. We then explore a little bit what kind of invariant these are (see Remark).

e. Consider the count mod 2 of intersections of a generic perturbation of a path with endpoints $A_\pm \in E \setminus E_*$ with E_*. Show that this is a well-defined count (mod 2) and it only depends on the homotopy class of the path relative to the endpoints. Note, moreover, that there is a unique path homotopy class, so this defines an invariant $\mu_{\mathbb{Z}/2}(A_+, A_-)$

   Hint: Use the transversality results from last problem for paths and homotopies of paths.

f. Given two invertible endomorphisms $A_\pm \in E \setminus E_*$, show that $\mu_{\mathbb{Z}/2}(A_+, A_-) = 0$ if and only if $\det A_+ \det A_- > 0$ (i.e. they have the same sign).

[6]By considering $A - \lambda$ for λ varying, this proposition is useful in tracking nearby eigenvalues for nearby matrices, which is part of the effort in the Spectral flow theorem.

g. * Consider the *signed* count of intersections of a generic perturbation of a path *in* E^s with endpoints in $E^s \setminus E^s_*$ with E^s_*. Show that this is a well-defined integer count and it only depends on the homotopy class of the path relative to the endpoints.

h. * State the spectral theorem from the lectures in the finite-dimensional setting. Use this to find two invertible symmetric matrices $A_\pm$ on $\mathbb{R}^2$ such that they can be connected via invertible matrices but not invertible and symmetric ones.

i. Bonus: Take $n = 2$ again. Show that E_1 is actually co-orientable[7]. If ν is a normal vector field along E_1 in E that is also tangent to E^s, then ν cannot match the canonical co-orientation of E^s_1 on every connected component.

   Hint: Using co-orientability of E_1 we can define a $\mathbb{Z}$-valued spectral flow for the pair (E, E_1). What would it mean for the orientations to always match in terms of these spectral flows?

Remark. Note that the $\mathbb{Z}/2$ spectral flow for (E, E_*) detects very little information. This is because the topology of the pair (E, E_*) is simple. One can even co-orient E_1 and define a signed count, but it only detects a tiny bit more of information. The topology of the pair (E^s, E^s_*) is much more subtle, the spectral flow for symmetric matrices is a much more interesting invariant (the last bonus problem this in opposition to (E, E^*_s)). In the lectures we have defined the spectral flow of symmetric operators (between two isomorphisms) and shown that it can be computed by the net change of eigenvalues. Choosing your favorite isomorphism (a base point), the spectral flow classifies with an integer the connected components (within the space of symmetric operators) of the space of such invertible operators.[8] This is a key way to define an *index*. We will see other instances of this.

Problem 3. The group of linear symplectomorphisms. A *symplectic vector space* (V, ω) is a vector space V equipped with a bilinear map $\omega : V \times V \to \mathbb{R}$ that is skew-symmetric ($\omega(u, v) = -\omega(v, u)$ for every $u, v \in V$) and non-degenerate (for every $u \in V$, if $\omega(u, v) = 0$ for every $v \in V$ then $u = 0$). We say that a subspace W of V is a *symplectic subspace* if $\omega|_W$ is non-degenerate. The *symplectic complement* of W is defined as $W^\omega = \{v \in V |\ \omega(v, w) = 0,\ \forall w \in W\}$.[9] We already know an example of symplectic vector space: $(\mathbb{R}^{2n}, \omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dy_i)$. On the first item we show that it is the only one.

a. Given any symplectic vector space (V, ω), there is a linear isomorphism $A : \mathbb{R}^{2n} \to V$ such that $A^*\omega = \omega_{\text{std}}$. Note that $A^*\omega(-, -) = \omega(A-, A-)$.

   Hint: Show the the existence of "symplectic basis": a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of V such that $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$ for $1 \le i \le j \le n$.

Given this, we consider $(\mathbb{R}^{2n}, \omega_{\text{std}})$ only. We define the linear symplectic group (or group of linear symplectomorphisms) as follows:

$$\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n, \mathbb{R}) :\ A^*\omega_{\text{std}} = \omega_{\text{std}}\}.$$

Note that the defining condition can be rephrased as $A^T J_0 A = J_0$ where

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

is the usual complex structure (after identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $(x, y) \longleftrightarrow x + iy$ for $x, y \in \mathbb{R}^n$).

[7] Can you show that E_{n-1} is co-orientable in general, without assuming $n = 2$?

[8] This is slightly wrong: techincally, we have defined the spectral flow in this generality but proven these things for the case of asymptotic operators.

[9] Note that one could also define W to be symplectic if and only if $W \oplus W^\omega = V$.Indeed, notice that W is symplectic if and only if $W \cap W^\omega = 0$. Notice too thar for any subspace W: $\dim W + \dim W^\omega = \dim V$, which follows from identifying W^ω with $W^\perp \subseteq V^*$ via the isomorphism $V \to V^*$ induced by ω. These two facts together show what we wanted.

b. * Show that if A and B are in $\mathrm{Sp}(2n)$, then so are A^T, A^{-1} and AB. Verify that $\det A = 1$ (so $\mathrm{Sp}(2n) \subseteq \mathrm{SL}(2n, \mathbb{R})$). Moreover, show that λ is an eigenvalue of A if and only if λ^{-1} is, in which case their multiplicities agree. Show as well that if ± 1 is an eigenvalue, it occurs with even multiplicity.

   Hint: Note that A^T and A^{-1} are similar.

c. Show that $\mathrm{Sp}(2n)$ is a Lie group of dimension $n(2n+1)$ with Lie algebra $\mathfrak{sp}(2n) = \{A \in \mathrm{End}(\mathbb{R}^{2n}) : A^T J_0 + J_0 A = 0\}$.[10] Note that $J_0\mathfrak{sp}(2n) = \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$.[11]

d. Prove the "2 out of 3" property[12]

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n).$$

   Hint: The first two equalities are easier than the last. For that, the following (you don't have to prove it) may be useful: block decomposing matrices by $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, we have that

$$\text{a matrix } A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \text{ is symplectic if and only if } A^{-1} = \begin{pmatrix} A_{22}^T & -A_{12}^T \\ -A_{21}^T & A_{11}^T \end{pmatrix}.$$

e. * Show that each $A \in \mathrm{Sp}(2n)$ decomposes uniquely as a product $A = UP$ for $U \in \mathrm{U}(n)$ and P a symmetric, positive-definite symplectic matrix.

   Hint: Make sense of $P = (A^T A)^{\frac{1}{2}}$ and $U = AP^{-1}$. To show that U is unitary, you may want to show that it is orthogonal and symplectic.

The above result is called symplectic polar decomposition. If we denote by $\mathrm{PDSS}(2n)$ the space of positive-definite symmetric and symplectic matrices on $\mathbb{R}^{2n}$, it shows that, in fact, $\mathrm{Sp}(2n) \cong \mathrm{U}(n) \times \mathrm{PDSS}(2n)$. We now show that $\mathrm{PDSS}(2n) \cong \mathbb{R}^{n(n+1)}$, which gives a deformation retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$, in particular it is connected.

f. Recall that a positive-definite symmetric matrix P can be uniquely written as $P = \exp(S)$ for S a real symmetric matrix. Show that P is symplectic if and only if

$$S = \begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

   where A and B are symmetric $n \times n$ matrices. Deduce from this that Show that $\mathrm{PDSS}(2n)$ is identified with $\mathrm{End}^{\mathrm{sym}}(\mathbb{R}^n) \times \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^n)$ and hence homeomorphic to $\mathbb{R}^{n(n+1)}$.

g. Show that the complex determinant $\det_{\mathbb{C}} : \mathrm{U}(n) \longrightarrow \mathbb{S}^1$ induces an isomorphism on fundamental groups. In particular, this also computes the fundamental group of $\mathrm{Sp}(2n)$.

   Hint: Consider the two fiber sequences $\mathrm{SU}(n) \to \mathrm{U}(n) \xrightarrow{\det_{\mathbb{C}}} \mathbb{S}^1$ and $\mathrm{SU}(n) \to \mathrm{SU}(n+1) \to \mathbb{S}^{2n+1}$.

h. Bonus: Use the symplectic polar decomposition to show that $\mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$.

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[10] Notice that this says that J_0 is skew-symmetric with respect to ω.

[11] I was asked about why there is a J in front of the symplectic connection ∇ in the definition of asymptotic operator (or why the symmetric matrix). That the Lie algebra of the symplectic group is the symmetric matrices multiplied by J_0 is the reason.

[12] Recall that the group of orthogonal matrices are defined by $A^T A = \mathbb{1}$, the complex general linear group by $AJ_0 = J_0 A$. The unitary group $\mathrm{U}(n)$ is given by those matrices $U = X + iY$ on $\mathbb{C}^n$ that satisfy $AA^* = \mathbb{1}$ for the conjugate transpose A^* of A.

Remark. This problem highlights the linear shadow of many important phenomena we study. That there is a unique symplectic vector space was what allowed us to use relative Moser to show Darboux's theorem. The intimate link of symplectic structures, almost-complex structures, Euclidean metrics, and Hermitian ones is very well encapsulated in the 2 out of 3 property, we will explore this further when we study the space of almost-complex structures more closely. The discussion on the topology of $\mathrm{Sp}(2n)$ will be crucial in characterizing the Conley-Zehnder index topologically/dynamically, by the intersection properties of the linearized Reeb flow. Particularly, exercises e. and g. are key in defining a topological monodromy angle, a mean count of the winding of the eigenvalues of the linearized flow. We will hopefully explore this further next week.