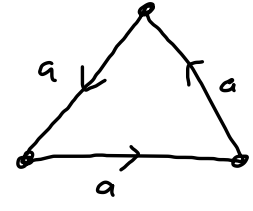
(1) Assume X is connected: this means that every continuous map $g: X \to \{0,1\}$ is constant. Now spse $Y \supseteq f(X) \xrightarrow{g} \{0,1\}$ is a continuous maps. It follows that $g \circ f : X \to \{0,1\}$ is also continuous, and therefore constant, and since f is surjective onto its image $f(X) \subseteq Y$, this implies that g is constant, $\Rightarrow f(X)$ is connected.

(2)(a) Let $\mathbb{R} \xrightarrow{q} \mathbb{R}/(0,1) = X$ denote the quotient projection; it sends $(0,1)$ to a single point $x_0 := q((0,1)) \in X$ and otherwise defines a bijection $\mathbb{R} \setminus (0,1) \xrightarrow{q} X \setminus \{x_0\}$. By the defn. of the quotient topology, a set $U \subseteq X$ is open $\iff$ $q^{-1}(U) \subseteq \mathbb{R}$ is open. Let $x := q(0)$ and $y := q(1)$. If $U \subseteq X$ is an open nbhd of x, then $q^{-1}(U) \subseteq \mathbb{R}$ is open and contains 0, thus it also contains some pts. in $(0,1) = q^{-1}(x_0)$; it follows that $(0,1) \subseteq q^{-1}(U)$ and $x_0 \in U$. By the same argument, every nbhd of y also contains x_0. (Comment: Another possible answer is $x := x_0$, $y := q(1)$, for instance.)

(2)(b) Finite sets are always compact, so let $K := \{x_0\} = q((0,1))$.

Then $q^{-1}(X \setminus \{x_0\}) = \mathbb{R} \setminus (0,1)$ is not open in $\mathbb{R}$, thus $X \setminus \{x_0\}$ is not open in X, implying that $\{x_0\}$ is not closed in X.

(Comment: This solution ignored the hint. Here's another solution using the hint: since $[0, \frac{1}{2}] \subseteq \mathbb{R}$ is compact, $K := q([0, \frac{1}{2}]) \subseteq X$ is also compact, but it is not closed because $q^{-1}(K) = [0,1)$ is not closed; equivalently, $q^{-1}(X \setminus K) = (-\infty, 0) \cup [1, \infty)$ is not open, $\Rightarrow X \setminus K \subseteq X$ is not open.)

(3) The equiv. rel. identifies each pt. on $\partial \mathbb{D}^2$ with 2 other pts. related to it by rotations of 120°. Using a homeo. of $\mathbb{D}^2$ to a triangle , we can identify X with space represented by the polygon 

.

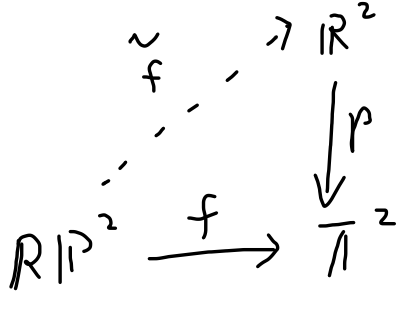
We used Seifert-van Kampen to compute π_1 in lecture for all spaces with such presentations; the answer in this case is

$$\pi_1(X) \cong \{ a \mid a^3 = e \} \cong \mathbb{Z}_3 .$$

(4) The universal cover of $\mathbb{T}^2$ is $\mathbb{R}^2 \xrightarrow{p} \mathbb{T}^2 = S^1 \times S^1$

$$(x,y) \longmapsto (e^{2\pi i x}, e^{2\pi i y}).$$

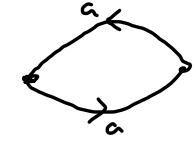
By the lifting theorem, the lifting problem



is solvable $\iff f_* \pi_1(\mathbb{RP}^2) \subseteq p_* \pi_1(\mathbb{R}^2) = 0$,

where both are regarded as subgroups of

$\pi_1(\mathbb{T}^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2$. (this was a homework exercise)

We have $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$ (derived from the presentation $\mathbb{RP}^2 \cong$ 

using an application of Seifert-van Kampen proved in lecture), and $\mathbb{Z}^2$ has no elements $x \in \mathbb{Z}^2$ satisfying $2x = 0, \Rightarrow \exists$ nontrivial homomorphisms

$\mathbb{Z}_2 \longrightarrow \mathbb{Z}^2$, $\Rightarrow$ $f_* \pi_1(\mathbb{RP}^2) = 0$ $\Rightarrow$ the lift $\tilde{f}$ exists.

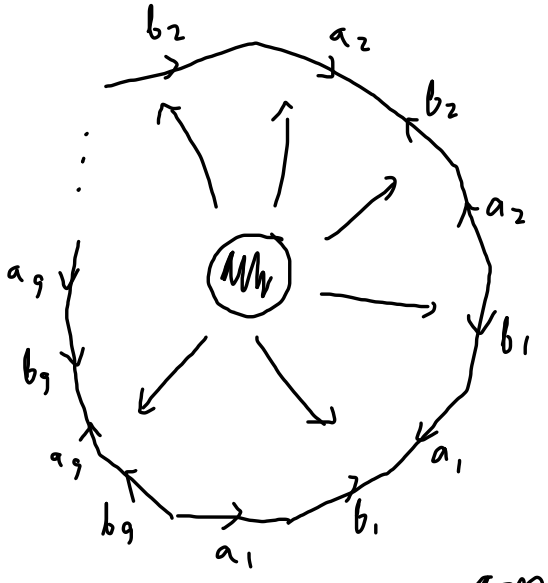
Since $\mathbb{R}^2$ is contractible, $\tilde{f} : \mathbb{RP}^2 \longrightarrow \mathbb{R}^2$ admits a homotopy $\{ \mathbb{RP}^2 \xrightarrow{\tilde{f}_t} \mathbb{R}^2 \}_{t \in [0,1]}$ to a constant map, and $\{ \mathbb{RP}^2 \xrightarrow{p \circ \tilde{f}_t} \mathbb{T}^2 \}_{t \in [0,1]}$ is then a htpy of f to a constant map.

(Comment: This was more justification than I'd consider strictly necessary — I would accept without proof the statement that since f lifts to a contractible cover, it is therefore homotopic to a constant.)

(5)(a) We use the fact (proved in an exercise) that $H_1(X;\mathbb{Z})$ for a path-conn. space X is iso. to the abelianization of $\pi_1(X)$.

We proved in lecture $\pi_1(\Sigma_g) \cong \left\{ a_1, b_1, \dots, a_g, b_g \;\middle|\; \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} = e \right\}$, and we also proved in an exercise that its abelianization is $\mathbb{Z}^{2g}$, thus $H_1(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$.

We proved in another exercise that $\pi_1(\Sigma_{g,1})$ is a free group with $2g$ generators; one can see this from the polygon picture



because cutting a hole out of the middle of the polygon makes it possible to defn. a deformation retraction to the boundary, which is (after dividing by the equivalence rel.) $\bigvee_{i=1}^{2g} S^1$, whose fundamental group (by an easy application of Seifert-van Kampen proved in lecture) is the free product of $2g$ copies of $\pi_1(S^1) \cong \mathbb{Z}$.

(Comment: This is also more than is strictly necessary — one could just cite the result of the exercise.)

By another result proved in a homework exercise, for 2 abelian groups G, H, the abelianization of the free product $G * H$ is $G \times H$. This extends by induction to any free product of fin.-many abelian groups, thus the abelianization of $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g}$ is $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{2g} = \mathbb{Z}^{2g}$.

We've proved: $H_1(\Sigma_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^{2g}$.

(5)(b) Arguing by contradiction, spse $h > g$, and $\exists$ a retraction

$$\Sigma_{g,1} \xrightarrow{f} A \subseteq \Sigma_{g,1} \quad \text{where} \quad A \cong \bigvee_{i=1}^{2h} S^1.$$

Since $\pi_1\left(\bigvee_{i=1}^{2h} S^1\right) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2h}$, we have

$$H_1(A;\mathbb{Z}) \cong \text{abelianization of } \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2h} \cong \mathbb{Z}^{2h}.$$

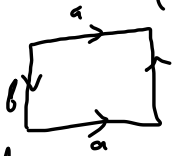
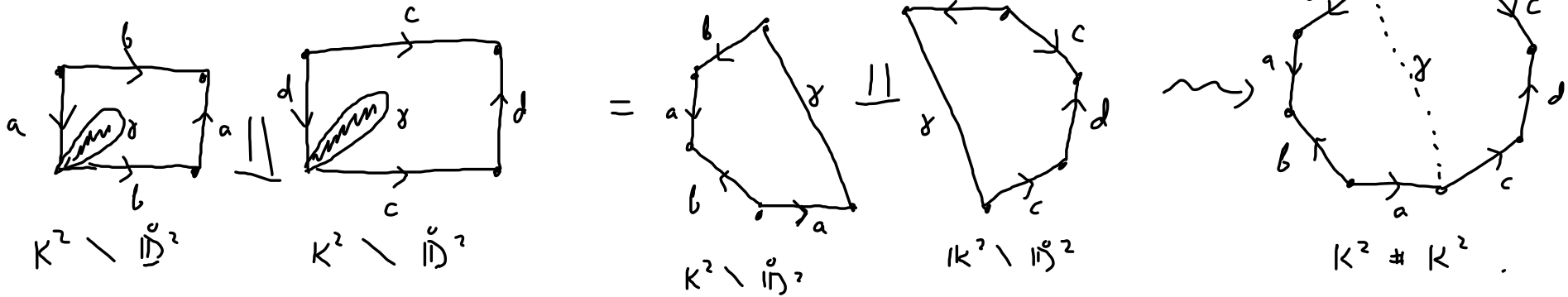
By a result proved in lecture, the hom. $H_1(\Sigma_{g,1};\mathbb{Z}) \xrightarrow{f_*} H_1(A;\mathbb{Z})$ induced by a retraction is surjective, so we have a surj. hom. $\mathbb{Z}^{2g} \longrightarrow \mathbb{Z}^{2h}$. Its image is generated by $2g < 2h$ elements and thus cannot be all of $\mathbb{Z}^{2h}$, so this is a contradiction.

(Comment: In part (b), I reused a few of the same results that had already been cited in part (a), without saying explicitly what they were. This is fine in context, since they had already been specified in part (a).)

(6) The surface is $\#_{k=1}^{4} \mathbb{RP}^2$.

Pf: We showed in lecture that the Klein bottle K^2 is $\mathbb{RP}^2 \# \mathbb{RP}^2$.

The result then follows from the claim: $\Sigma \cong K^2 \# K^2$.

Indeed, the following picture demonstrates how to construct Σ by cutting holes out of 2 Klein bottles $K^2 =$ 

and then gluing them together: