PRACTICE EXAM (PROBEKLAUSUR)

Instructions

You have three hours in total, though the exam is designed to be doable in significantly less time than that. For reference, you may use any notes or books that you bring with you, but nothing electronic, i.e. no calculators or smartphones.

All answers require justification (within reason) in order to receive full credit, though you need not reprove any results that were proved in the lectures or on the problem sets. Keep in mind that if you get stuck on one part of a problem, it may sometimes be possible to skip it and do the next part.

Disclaimer: *I have not carefully timed this practice exam, so it is probably somewhat longer than the actual exam will be.*

Problems [100 pts total]

1. Recall that on $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, the standard spherical coordinates (θ, ϕ) are defined via the relations

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta.$$

Outside of the two poles $p_\pm := (0, 0, \pm 1) \in S^2$, these relations determine values of $\theta \in (0, \pi)$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ uniquely, while $\theta \in \{0, \pi\}$ is also well defined at $p_\pm$, but ϕ is not.

   (a) [6 pts] Choose two open subsets $\mathcal{U}_1, \mathcal{U}_2 \subset S^2$ that cover $S^2 \setminus \{p_+, p_-\}$ such that the coordinates (θ, ϕ) can be viewed as defining two charts $x_j : \mathcal{U}_j \to \mathbb{R}^2$ for $j = 1, 2$. Specify the image of both charts, and write down the transition map $x_2 \circ x_1^{-1} : x_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to x_2(\mathcal{U}_1 \cap \mathcal{U}_2)$. Is it orientation preserving?

   (b) [6 pts] Define charts for S^2 on neighborhoods of the poles $p_\pm$ such that their transition maps with the two charts from part (a) are smooth and orientation preserving. This gives S^2 the structure of an oriented smooth 2-manifold.

   (c) [8 pts] Prove that there is a well-defined 2-form on S^2 which takes the form $\omega = \sin\theta \, d\theta \wedge d\phi$ in either of the charts from part (a), and show that it is a volume form everywhere (including at the poles). What does it look like in the charts from part (b)?

2. Suppose M is a smooth n-manifold, Σ is a closed 2-manifold and $\omega \in \Omega^2(M)$ is a smooth 2-form on M.

   (a) [10 pts] Show that if ω is closed, then for any two maps $f, g : \Sigma \to M$ that are smoothly homotopic, $\int_\Sigma f^*\omega = \int_\Sigma g^*\omega$.
   Hint: There are at least two possible approaches. You could use the theorem that homotopic maps induce the same pullback on de Rham cohomology, but you could also do without this and use only Stokes' theorem.

   (b) [10 pts] Assume $M = \mathbb{R}^{2n}$ with coordinates $(q^1, p^1, \dots, q^n, p^n)$, and ω is the standard symplectic form $\sum_{j=1}^n dp^j \wedge dq^j$. Show that $\int_\Sigma f^*\omega$ is then zero for all smooth maps $f : \Sigma \to \mathbb{R}^{2n}$.

3. Given a constant $c \in \mathbb{R}$, let $\xi_c \subset T\mathbb{R}^3$ denote the smooth distribution on $\mathbb{R}^3$ defined as the kernel of the 1-form $\lambda_c := dz + cx \, dy$ in standard (x, y, z) coordinates.

   (a) [10 pts] For which values of c is ξ_c an integrable distribution? When it is integrable, what do its integral submanifolds look like?
   Hint: Show that there exist unique functions $A, B : \mathbb{R}^3 \to \mathbb{R}$ such that the vector fields $X := \partial_x + A\partial_z$ and $Y := \partial_y + B\partial_z$ are everywhere tangent to ξ_c. What can you then say about $d\lambda_c(X, Y)$, and how does this relate to the Frobenius integrability theorem?

(b) [5 pts] Show that for every smoothly immersed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, there exists a smooth function $f : \mathbb{R} \to \mathbb{R}$, uniquely defined up to addition of a constant, such that the curve $\tilde{\gamma} := (\gamma, f) : \mathbb{R} \to \mathbb{R}^3$ is everywhere tangent to the distribution ξ_1.

(c) [5 pts] If the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ in part (b) is 1-periodic, meaning $\gamma(t+1) = \gamma(t)$ for all t, under what circumstances will its lift $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^3$ also be 1-periodic? Show that this can never happen if $\gamma : [0,1] \to \mathbb{R}^2$ is an embedded parametrization of the boundary of a compact region in $\mathbb{R}^2$.

4. Assume G is a connected n-dimensional Lie group with Lie algebra $\mathfrak{g} = T_eG$. As we showed in one of the problem sets this semester, the tangent bundle of G is necessarily trivializable, therefore G is orientable.

   (a) [10 pts] For which values of n is the diffeomorphism $G \to G : g \mapsto g^{-1}$ orientation preserving/reversing?

   (b) [10 pts] Show that for any $g \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$,

   $$\exp(tX)g = \varphi^t_{X^R}(g),$$

   where X^R denotes the unique right-invariant vector field on G with $X^R(e) = X$, and $\varphi^t_{X^R} : G \to G$ is its flow for time t.

5. [10 pts] Consider the smooth function

   $$F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R} : (x, y, z) \mapsto x^2 + y^2 - z^2.$$

   Since we've removed the origin from the domain, F has no critical points and the level sets $M_c := F^{-1}(c)$ for $c \in \mathbb{R}$ are all smooth submanifolds of $\mathbb{R}^3 \setminus \{0\}$. Introduce on each of these surfaces the Riemannian metric induced from the standard inner product on $\mathbb{R}^3$ via the embedding $M_c \hookrightarrow \mathbb{R}^3$. For which values of c does M_c have positive/negative/zero Gaussian curvature? You don't need to compute it explicitly, but justify your answer.

6. [10 pts] Suppose $E \xrightarrow{\pi} M$ is a smooth vector bundle of rank $m \in \mathbb{N}$ over the field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$). Show that if $E \xrightarrow{\pi} M$ admits a flat connection, then there exists an open covering $M = \bigcup_\alpha \mathcal{U}_\alpha$, together with a collection of local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ such that the transition functions $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{F}^{m \times m}$ are all constant.
   Hint: If parallel sections always exist locally, then you can use them to construct local frames.