PROBLEM SET 11

Suggested reading

Lecture notes (on the website): still Chapter 4 (we'll get to Chapter 5 next week)

Problems

1. One of the standard examples of "non-Euclidean" geometry is a Riemannian manifold known as the *Poincaré half-plane* $(\mathbb{H}, h)$. It is defined as the 2-manifold

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

equipped with the Riemannian metric

$$h = \frac{1}{y^2} g_E,$$

where g_E is the standard Euclidean metric on $\mathbb{R}^2$. In other words, the inner product of two vectors $X, Y \in T_{(x,y)}\mathbb{H}$ tangent at the point $(x, y) \in \mathbb{H}$ is defined as

$$h(X, Y) = \frac{1}{y^2} \langle X, Y \rangle,$$

where $\langle \, , \, \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^2$ (we are using the canonical identification of $T_{(x,y)}\mathbb{H}$ with $\mathbb{R}^2$).

(a) Show that a smooth path $\gamma(t) = (x(t), y(t)) \in \mathbb{H}$ is a geodesic on $(\mathbb{H}, h)$ if and only if it satisfies the following second-order system of ordinary differential equations:

$$\begin{aligned} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= 0 \\ \ddot{y} + \frac{1}{y}\left(\dot{x}^2 - \dot{y}^2\right) &= 0. \end{aligned} \tag{1}$$

Hint: $\mathbb{H}$ has an obvious global chart, so this is a straightforward computation in coordinates if you remember the relevant formulas. Specifically, the geodesic equation in coordinates $(x^1, \ldots, x^n)$ generally takes the form

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0,$$

where the Christoffel symbols for the Levi-Civita connection with respect to a Riemannian metric g are determined by its components $g_{ij} = g(\partial_i, \partial_j)$ and the associated inverse matrix (with entries denoted by g^{ij}) according to

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\partial_j g_{k\ell} + \partial_k g_{\ell j} - \partial_\ell g_{jk}\right).$$

(The latter was derived in our proof of existence and uniqueness of the Levi-Civita connection; see also §4.3.3 in the lecture notes.)

(b) Show that for any constants $x_0 \in \mathbb{R}$ and $r > 0$, Equations (1) admit solutions of the form

$$(x(t), y(t)) = (x_0, y(t))$$

for some function $y(t) > 0$, as well as

$$(x(t), y(t)) = (x_0 + r\cos\theta(t), r\sin\theta(t)).$$

for some function $\theta(t) \in (0, \pi)$.

(c) Prove that the solutions of part (a) give *all* geodesics on $(\mathbb{H}, h)$, and that any two points in $\mathbb{H}$ can be joined by a *unique* geodesic. *Note: You can prove this mostly with pictures.*

(d) Compute the length of the geodesic segment joining (x_0, y_0) and (x_0, y_1) for any $0 < y_0 < y_1$. Compute also the length of the geodesic segment joining $(x_0 + r\cos\theta_0, r\sin\theta_0)$ and $(x_0 + r\cos\theta_1, r\sin\theta_1)$ for any $0 < \theta_0 < \theta_1 < \pi$. Use these results to show that for all $p \in \mathbb{H}$ and $X \in T_p\mathbb{H}$, the geodesic $t \mapsto \exp(tX)$ exists for all $t \in \mathbb{R}$. (Riemannian manifolds with this property are called *geodesically complete.*)

2. Assume M is a smooth n-manifold with a submanifold $\Sigma \subset M$ of dimension $m < n$. At each point $p \in \Sigma$, the tangent space $T_p\Sigma$ is then naturally a linear subspace of T_pM, so one can define the *normal bundle* of Σ as a rank $n - m$ vector bundle $N_{\Sigma/M} \to \Sigma$ whose fiber for each $p \in \Sigma$ is

$$(N_{\Sigma/M})_p = T_pM / T_p\Sigma.$$

(Take a moment before continuing to consider how one might prove that $N_{\Sigma/M}$ is a smooth vector bundle.)

(a) Show that for any Riemannian metric g on M, $N_{\Sigma/M}$ is isomorphic to the subbundle of $TM|_\Sigma$ whose fiber at each $p \in \Sigma$ is the orthogonal complement of $T_p\Sigma$ in T_pM with respect to g.

(b) Use the inverse function theorem to prove the *tubular neighborhood theorem*: there exists a neighborhood $\mathcal{U} \subset N_{\Sigma/M}$ of the zero-section and an embedding $\Phi : \mathcal{U} \hookrightarrow M$ whose image is a neighborhood of Σ in M such that, identifying Σ with the zero-section of $N_{\Sigma/M}$, the restriction of Φ to the zero-section is just the inclusion of Σ into M.
Hint: Choose a Riemannian metric so that $N_{\Sigma/M}$ can be identified as in part (a) with a subbundle of $TM|_\Sigma$, then use the exponential map.

The tubular neighborhood theorem looks a bit abstract in its general form, but notice what it implies if we also assume that Σ is compact and its normal bundle happens to be trivial: it then identifies a neighborhood of Σ in M with $\Sigma \times \mathbb{D}^{n-m}$ such that Σ becomes $\Sigma \times \{0\}$; here $\mathbb{D}^{n-m}$ denotes the $(n - m)$-dimensional unit disk.

3. Suppose M is a smooth manifold, ∇ is a connection on its tangent bundle, $H(TM) \subset T(TM)$ denotes the associated horizontal subbundle and $\mathrm{Hor}_v : T_pM \to H_v(TM)$ is the corresponding horizontal lift isomorphism defined for each $p \in M$ and $v \in T_pM$. This allows us to define a vector field X on the total space TM by

$$X(v) := \mathrm{Hor}_v(v).$$

(a) How is the flow of this vector field related to the geodesic equation on M with respect to the connection ∇?

(b) Show that if M is a closed manifold and ∇ is the Levi-Civita connection with respect to a Riemannian metric g, then (M, g) is geodesically complete (cf. Problem 1(d)). Prove this as a corollary of the fact that flows of vector fields on closed manifolds exist for all time.
Hint: While TM itself is not compact, all flow lines of X are confined to certain compact submanifolds—explain.

(c) It is perfectly possible for a noncompact Riemannian manifold to be geodesically complete, e.g. this is true for $\mathbb{R}^n$ with its standard Euclidean metric. However, show that $\mathbb{R}^n$ also admits Riemannian metrics that are not geodesically complete.
Hint: $\mathbb{R}^n$ is diffeomorphic to the open unit ball.

4. Recall that a *pseudo-Riemannian* metric is a tensor field $g \in \Gamma(T_2^0M)$ that is everywhere symmetric and nondegenerate, but not necessarily positive-definite (see Problem Set 9 #2). A large portion—but not all—of standard Riemannian geometry extends to the pseudo-Riemannian case. Show in particular that the notion of the Levi-Civita connection generalizes to this context, i.e. for every pseudo-Riemannian metric g, TM admits a unique symmetric connection ∇ that is compatible with g in the sense that $\nabla g \equiv 0$. Similarly, geodesics γ with respect to ∇ have constant "speed squared" $g(\dot\gamma, \dot\gamma)$—which may be positive, zero, or negative—and they are critical points of the energy functional.

Can you think of any results that do *not* obviously extend to the pseudo-Riemannian setting?