PROBLEM SET 12

Suggested reading

Lecture notes (on the website): Chapter 5

Problems

1. In this problem we shall prove the Gauss lemma, which states that for any point p in a Riemannian manifold (M, g) and any tangent vector $X \in T_pM$ with $r_0 := |X| := \sqrt{g(X,X)} > 0$ sufficiently small, the geodesic segment $[0,1] \to M : t \mapsto \exp(tX)$ meets each of the spheres

$$S(r) := \left\{ \exp(Y) \in M \;\middle|\; Y \in T_pM \text{ with } |Y| = r \right\}$$

around p orthogonally. We used this in lecture to prove that the unique short geodesic between two nearby points is always the *shortest* path between those points.

We will again use the notion of "n-dimensional polar coordinates," which can be defined as follows. Abbreviate $\langle X, Y \rangle := g(X,Y)$ and $|X| := \sqrt{\langle X, X \rangle}$ as usual, and choose any diffeomorphism[1]

$$\psi : S^{n-1} \to ST_pM := \left\{ Y \in T_pM \;\middle|\; |Y| = 1 \right\}.$$

Then for some small $r_0 > 0$, we consider the smooth map

$$\Phi : [0, r_0] \times S^{n-1} \to M : (r, x) \mapsto \exp(r\psi(x)),$$

whose image we shall denote by $\mathcal{U} \subset M$. Since exp maps a neighborhood of 0 in T_pM diffeomorphically to a neighborhood of p in M, we can assume if r_0 is sufficiently small that Φ restricts to a diffeomorphism

$$(0, r_0] \times S^{n-1} \xrightarrow{\Phi} \mathcal{U} \setminus \{p\},$$

which identifies the sphere $\{r\} \times S^{n-1}$ with $S(r)$ for each $r \in (0, r_0]$. In the case $n = 2$, the diffeomorphism $(0, r_0] \times S^1 \to \mathcal{U} \setminus \{p\}$ determines two polar coordinate vector fields ∂_r and ∂_θ, and the geodesics emanating from p are precisely the radial paths in these coordinates, so the Gauss lemma amounts to the statement that $\langle \partial_r, \partial_\theta \rangle \equiv 0$.

In the general case, we shall denote by ∂_r the vector field on $\mathcal{U} \setminus \{p\}$ which associates to each point $q = \Phi(r, x)$ the tangent vector $\frac{\partial}{\partial r}\Phi(r, x) \in T_qM$. Similarly, any vector field $Y \in \operatorname{Vec}(S^{n-1})$ gives rise to a vector field Z on $\mathcal{U} \setminus \{p\}$ which associates to each point $q = \Phi(r, x)$ the vector

$$Z(q) := T\Phi(0, Y(x)) \in T_qM,$$

where we view $(0, Y(x))$ as an element of $T_{(r,x)}\left([0, r_0] \times S^{n-1}\right)$ via its natural isomorphism with $T_r[0, r_0] \times T_xS^{n-1} = \mathbb{R} \times T_xS^{n-1}$. The vector field Z is a generalization of ∂_θ from the $n = 2$ case; since $Y \in \operatorname{Vec}(S^{n-1})$ can take any value we like at a given point, it will now suffice to prove that $\langle \partial_r, Z \rangle$ must vanish identically on $\mathcal{U} \setminus \{p\}$.

   (a) Show that $\langle \partial_r, \partial_r \rangle \equiv 1$.
   Hint: Notice that the paths $t \mapsto \Phi(t, x)$ for each $x \in S^{n-1}$ are geodesics and therefore have constant speed. What is their speed at $t = 0$?
   Remark: One can deduce from this that ∂_r, while a smooth vector field on $\mathcal{U} \setminus \{p\}$, does not extend continuously over the point p. (Why not?) This is a symptom of the fact that polar coordinates are singular at $r = 0$.

[1] We would normally define $\psi : S^{n-1} \to ST_pM$ by choosing an orthonormal basis of T_pM, thus identifying T_pM with $\mathbb{R}^n$ so that ST_pM matches the standard unit sphere. For the proof at hand, it really doesn't matter.

(b) Show that in contrast to ∂_r, the vector field $Z \in \mathrm{Vec}(\mathcal{U} \setminus \{p\})$ admits a continuous extension over $\mathcal{U}$, with $Z(p) = 0$. *Hint: What does Φ do to the sphere $\{0\} \times S^{n-1}$?*

(c) Assuming ∇ is the Levi-Civita connection on (M, g), show that ∂_r and $\nabla_Z(\partial_r)$ are orthogonal everywhere on $\mathcal{U} \setminus \{p\}$. *Hint: Differentiate $\langle \partial_r, \partial_r \rangle$ in the direction of Z.*

(d) Show that $[\partial_r, Z] \equiv 0$. (Do their flows commute?)

(e) Show that $\mathcal{L}_{\partial_r} \langle \partial_r, Z \rangle \equiv 0$.
Hint: You will need to use both of the defining properties of the Levi-Civita connection here, i.e. symmetry, and compatibility with the metric.

(f) Show that the real-valued function $\langle \partial_r, Z \rangle$ on $\mathcal{U} \setminus \{p\}$ has a continuous extension over $\mathcal{U}$ that equals 0 at p, and conclude via the previous steps that it vanishes identically. The Gauss lemma is thus proven. *Hint: Remember the Cauchy-Schwarz inequality for inner products: $|\langle v, w \rangle| \le |v| \cdot |w|$.*

2. A diffeomorphism $\varphi : M \to N$ between two Riemannian manifolds (M, g) and (N, h) is called an *isometry* if $\varphi^* h = g$. This has the following geometric interpretation: φ is an isometry if and only if its tangent map preserves the lengths of tangent vectors and angles between them (as measured in terms of the metrics g and h).

(a) Show that if $\varphi : (M, g) \to (N, h)$ is an isometry, then a curve $\gamma : (a, b) \to M$ is a geodesic with respect to g if and only if $\varphi \circ \gamma : (a, b) \to N$ is a geodesic with respect to h.
Remark: This is the kind of statement that you will probably regard as "obvious" once you have developed a certain comfort level with differential geometry, but the first time you see it, it's worth thinking a little about why it is true. The proof is not hard.

Now suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and identity element $e \in G$, let $L_g : G \to G : h \mapsto gh$ and $R_g : G \to G : h \mapsto hg$ denote the diffeomorphisms defined by left- and right-translation respectively, and given any $X \in \mathfrak{g}$, denote by $X^L, X^R \in \mathrm{Vec}(G)$ the left- and right-invariant vector fields respectively satisfying $X^L(e) = X^R(e) = X$. Consider the diffeomorphism

$$\mathrm{inv} : G \to G : g \mapsto g^{-1}.$$

(b) Show that for any $X \in \mathfrak{g}$ and $g \in G$,

$$T(\mathrm{inv})(X^L(g)) = -X^R(g^{-1}).$$

Hint: First prove it assuming $g = e$, then for the general case, write $X^L(g)$ and $X^R(g^{-1})$ in terms of X and the maps L_g and $R_{g^{-1}}$.

A Riemannian metric $\langle\ ,\ \rangle$ on G is called *left-invariant* if $L_g : G \to G$ is an isometry from $(G, \langle\ ,\ \rangle)$ to itself for every $g \in G$. One similarly defines right-invariant metrics using the maps R_g, and a metric that is both left- and right-invariant is called *bi-invariant.* One can use an averaging procedure (via the bi-invariant volume forms of Problem Set 8 #3) to show that every *compact* Lie group admits a bi-invariant Riemannian metric.

(c) Show that for any bi-invariant metric $\langle\ ,\ \rangle$ on G, $\mathrm{inv} : G \to G$ is an isometry of $(G, \langle\ ,\ \rangle)$ to itself.

(d) Deduce that if $\langle\ ,\ \rangle$ is bi-invariant, then every geodesic $\gamma : (-\epsilon, \epsilon) \to G$ with $\gamma(0) = e$ satisfies

$$\gamma(-t) = (\gamma(t))^{-1} \quad \text{for all} \quad t \in \mathbb{R}.$$

One can go further and show that whenever $\langle\ ,\ \rangle$ is bi-invariant, the geodesics in part (d) extend to smooth maps $\mathbb{R} \to G$ which are also group homomorphisms. This means that in this setting, our two definitions for the map $\exp : T_e G \to G$, one in terms of group homomorphisms $\mathbb{R} \to G$ and the other via geodesics, coincide. To this end:

(e) Given any left-invariant Riemannian metric $\langle\ ,\ \rangle$ on a Lie group G, show that the following two conditions are equivalent:

i. Every smooth group homomorphism $\mathbb{R} \to G$ is a geodesic;

ii. Every pair of left-invariant vector fields $X, Y \in \mathrm{Vec}(M)$ satisfies

$$\nabla_X Y = \frac{1}{2}[X, Y], \tag{1}$$

where ∇ is the Levi-Civita connection for $\langle\ ,\ \rangle$.

Hint 1: Recall that smooth group homomorphisms $\mathbb{R} \to G$ are also flow lines of left-invariant vector fields.

Hint 2: Given condition (i), what can you say about $\nabla_{X+Y}(X+Y)$ if X and Y are both left invariant?

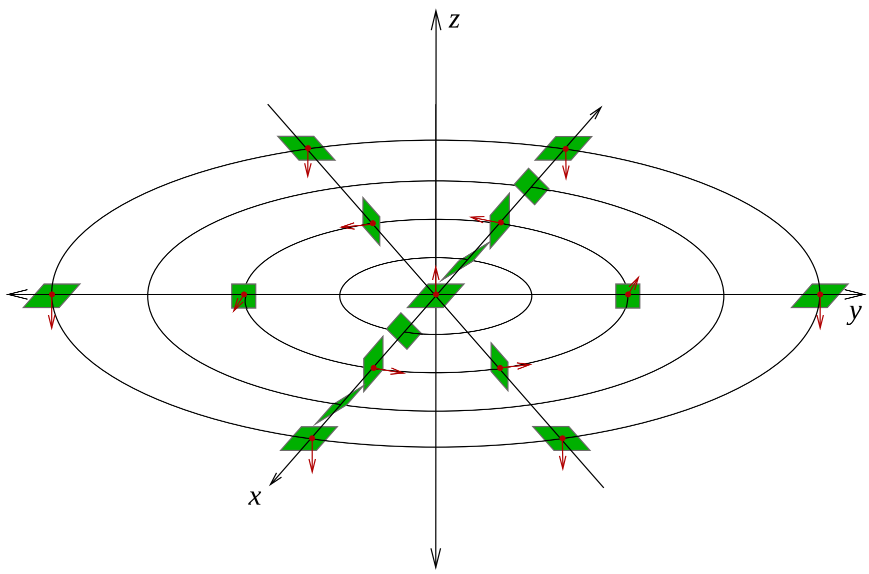
It turns out that Equation (1) *holds whenever $\langle\ ,\ \rangle$ is bi-invariant, though the proof of this is a bit tedious, so we will not work through it here. This has the following interesting consequence. By a standard result in Riemannian geometry called the Hopf-Rinow theorem, if (M, g) is a connected Riemannian manifold containing a point p for which all geodesics through p exist for all time, then any two points on M can be joined by a geodesic. It follows that for every compact and connected Lie group G, the algebraic exponential map* $\exp : \mathfrak{g} \to G$ *(defined in terms of group homomorphisms $\mathbb{R} \to G$) is surjective. This sometimes also holds for noncompact groups, but we saw in Problem Set 8 #5 that it does not hold for* $\mathrm{SL}(2, \mathbb{R})$*; in particular, one deduces from this that* $\mathrm{SL}(2, \mathbb{R})$ *does not admit a bi-invariant Riemannian metric.*

3. The following lemma is needed for our proof of the Frobenius integrability theorem in lecture: if $\xi \subset TM$ is a smooth k-dimensional distribution on an n-manifold M, then for every $p \in M$, there exists a neighborhood $\mathcal{U} \subset M$ of p, a smooth k-dimensional manifold Q and a smooth map $\pi : \mathcal{U} \to Q$ which is a fiber bundle admitting a connection whose horizontal subbundle is ξ. Prove this.

   Hint: The neighborhood $\mathcal{U}$ can be arbitrarily small, so you should be able to define a chart on $\mathcal{U}$ and write down π in coordinates. Don't be too clever; choose $\mathcal{U}$ and Q to be as simple as possible.

4. Suppose M is an oriented 3-manifold and $\lambda \in \Omega^1(M)$ is nowhere zero, i.e. for all $p \in M$ there exist vectors $X \in T_pM$ with $\lambda(X) \neq 0$. Then at every $p \in M$, the *kernel* $\ker \lambda_p = \{X \in T_pM \mid \lambda(X) = 0\}$ is a 2-dimensional subspace of T_pM, and the union of these for all p defines a smooth 2-dimensional distribution

$$\xi := \ker \lambda \subset TM.$$



   (a) Show that the following conditions are equivalent:

i. $\lambda \wedge d\lambda \equiv 0$;

ii. For all $p \in M$ and $X, Y \in T_p M$, $d\lambda(X, Y) = 0$;

iii. ξ is integrable.

Hint: In Problem Set 6 #4(c), you will find a useful formula for $d\lambda$ as a C^∞-bilinear form on vector fields. Combine this with the Frobenius theorem.

The 1-form λ is called a *contact form* if $\lambda \wedge d\lambda$ is a volume form; the distribution ξ (called the *contact structure*) is then "as non-integrable as possible." An example on $\mathbb{R}^3$ is shown in the figure above. Such examples can be constructed by the following trick. Let (ρ, ϕ, z) denote the standard cylindrical coordinates on $\mathbb{R}^3$, so $x = \rho \cos \phi$ and $y = \rho \sin \phi$. Choose smooth real-valued functions $f(\rho), g(\rho)$ and define λ at (ρ, ϕ, z) by

$$\lambda = f(\rho)\ dz + g(\rho)\ d\phi. \tag{2}$$

(b) Since the coordinates (ρ, ϕ, z) are not well defined at $\rho = 0$, there is of course some danger that the 1-form defined in Equation (2) might be singular at the z-axis. Show that λ is in fact smooth on all of $\mathbb{R}^3$ and satisfies $\lambda \wedge d\lambda \neq 0$ near the z-axis if we assume $f(\rho) = 1$ and $g(\rho) = \rho^2$ for ρ sufficiently close to 0. *Hint: Convert to Cartesian coordinates.*

(c) Assuming f and g take the form desribed above for ρ near 0, show that λ is a contact form if and only if

$$f(\rho)g'(\rho) - f'(\rho)g(\rho) \neq 0$$

for all $\rho > 0$. What does this mean geometrically about the curve $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$? Interpret this in terms of "twisting" of the planes ξ_p as $p \in \mathbb{R}^3$ moves along radial paths away from the z-axis.

(d) By a fundamental result in contact geometry known as *Gray's theorem*, contact structures have the following remarkable "stability" property: if M is a closed 3-manifold and $\{\xi_t\}_{t \in [0,1]}$ is any smooth family of contact structures on M, then they are all "equivalent" in the sense that there exists a smooth family of diffeomorphisms $\{\varphi_t : M \to M\}_{t \in [0,1]}$ such that $\varphi_0 = \mathrm{Id}$ and $T\varphi_t(\xi_0) = \xi_t$ for all $t \in [0, 1]$. Show that this is not true in general for arbitrary smooth families of distributions, e.g. it becomes false if we assume that ξ_0 is integrable but ξ_t is a contact structure for each $t > 0$.