PROBLEM SET 13

Suggested reading

Lecture notes (on the website): Chapter 6, up to §6.3.1

Problems

1. Recall that in Problem Set 11 #1, we computed the geodesics on the *Poincaré half-plane* $(\mathbb{H}, h)$, defined as $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the Riemannian metric $h = \frac{1}{y^2} g_E$, where g_E denotes the standard Euclidean metric on $\mathbb{R}^2$.

   (a) Write down the natural volume form on $\mathbb{H}$ determined by the metric h, i.e. the unique 2-form that evaluates to 1 on any positively oriented orthonormal basis. Show that with respect to this volume form, any region of the form $[a, b] \times [c, \infty) \subset \mathbb{H}$ for $-\infty < a < b < \infty$ and $c > 0$ has finite area, while regions of the form $[a, b] \times (0, c] \subset \mathbb{H}$ have infinite area.

   (b) By drawing pictures, show that the sum of the angles in a geodesic triangle in $(\mathbb{H}, h)$ can be arbitrarily small. (By "geodesic triangle" we mean a compact region in $\mathbb{H}$ bounded by three geodesic segments.)

   (c) Compute all components (with respect to the obvious coordinates) of the Riemann curvature tensor for the Levi-Civita connection on $(\mathbb{H}, h)$.

   (d) Compute the Gaussian curvature of $(\mathbb{H}, h)$.
   Hint: The answer should be a negative constant. Why is this consistent with part (b)?

2. An *isometry* of a Riemannian manifold (M, g) is a diffeomorphism $\varphi : M \to M$ such that $\varphi^* g = g$. The isometries of (M, g) form a topological group $\mathrm{Isom}(M, g)$. Its structure in a neighborhood of the identity map can be understood by considering smooth 1-parameter families $\varphi_t \in \mathrm{Isom}(M, g)$ with $\varphi_0 = \mathrm{Id}$. In particular, differentiating this with respect to t at $t = 0$ gives a vector field

$$X(p) = \frac{d}{dt}\varphi_t(p)\Big|_{t=0},$$

   which must satisfy $\mathcal{L}_X g \equiv 0$ due to the condition $\varphi_t^* g = g$. A vector field satisfying this condition is called a *Killing vector field*. Intuitively, we think of it as an "infinitessimal isometry".

   (a) Show that if ∇ is any symmetric connection on $TM \to M$, $X \in \mathrm{Vec}(M)$, $\lambda \in \Omega^1(M)$ and $Y \in TM$, then $(\mathcal{L}_X \lambda)(Y) = (\nabla_X \lambda)(Y) + \lambda(\nabla_Y X)$.
   Hint: Choose a smooth map $\alpha(s, t) \in M$ defined for $(s, t) \in \mathbb{R}^2$ near the origin such that $\partial_s \alpha(s, t) = X(\alpha(s, t))$ and $\partial_t \alpha(0, 0) = Y$. It will be crucial that the connection is symmetric, so $\nabla_s \partial_t \alpha = \nabla_t \partial_s \alpha$.

   (b) Generalize the above result to the formula

$$\begin{aligned}(\mathcal{L}_X T)(Y_1, \dots, Y_k) = (\nabla_X T)(Y_1, \dots, Y_k) &+ T(\nabla_{Y_1} X, Y_2, \dots, Y_k) \\ &+ T(Y_1, \nabla_{Y_2} X, \dots, Y_k) + \dots + T(Y_1, \dots, Y_{k-1}, \nabla_{Y_k} X),\end{aligned}$$

   valid for any covariant tensor field $T \in \Gamma(T^0_k M)$.

   (c) Applying the formula above with the Levi-Civita connection so that $\nabla g \equiv 0$, we find $\mathcal{L}_X g \equiv 0$ if and only if

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

   for all $p \in M$ and $Y, Z \in T_p M$. This is called the *Killing equation*.

The bundle metric on $TM \to M$ defines for each $p \in M$ a so-called *musical isomorphism*

$$\flat : T_pM \to T_p^*M : Y \mapsto Y^\flat$$
$$Y^\flat(Z) := g(Y, Z).$$

Thus a vector field $X \in \mathrm{Vec}(M)$ gives rise to a 1-form $X^\flat \in \Omega^1(M)$, and this is a one-to-one correspondence. Show that for any $X \in \mathrm{Vec}(M)$ and $Y \in TM$,

$$(\nabla_Y X)^\flat = \nabla_Y (X^\flat).$$

Then show that X satisfies the Killing equation if and only if the tensor field $\nabla X^\flat \in \Gamma(T_2^0 M)$ defined by $\nabla X^\flat(Y, Z) := (\nabla_Y X^\flat)(Z)$ is antisymmetric.

(d) By the above result, solving the Killing equation is equivalent to finding a 1-form $\lambda \in \Omega^1(M)$ such that

$$\nabla\lambda(Y, Z) + \nabla\lambda(Z, Y) = 0. \tag{1}$$

Suppose $\gamma(s) \in M$ is a geodesic through $\gamma(0) = p \in M$. Show that if $\lambda \in \Omega^1(M)$ satisfies Equation (1), then as a section of T^*M along γ, it also satisfies the second order linear differential equation

$$\nabla_s^2 \lambda = \lambda(R(\dot\gamma, \cdot)\dot\gamma), \tag{2}$$

or to be more precise, for any $Y \in T_{\gamma(s)}M$, $(\nabla_s \nabla_s \lambda)(Y) = \lambda(R(\dot\gamma(s), Y)\dot\gamma(s))$. Here $R(X, Y)Z$ denotes the Riemann curvature tensor $R : TM \oplus TM \oplus TM \to TM$ defined by the Levi-Civita connection on $TM \to M$.

Hint: This is tricky, but here are some tips to get you started. If $Y(s) \in T_{\gamma(s)}M$ is a parallel *vector field along γ, then show that $(\nabla_s^2 \lambda)(Y) = \partial_s^2 (\lambda(Y))$. One can extend $\gamma(s)$ to a smooth map $\alpha(s, t)$ with $\alpha(s, 0) = \gamma(s)$ so that $\partial_t \alpha(s, 0) = Y(s)$. Then in terms of covariant partial derivatives, Equation (1) says*

$$(\nabla_s \lambda)(\partial_t \alpha(s, t)) + (\nabla_t \lambda)(\partial_s \alpha(s, t)) = 0.$$

The rest follows from intelligent use of commuting (or non-commuting) partial derivatives, including the symmetry of the connection and the definition of the curvature tensor.

(e) We now appeal to a general fact about second order linear differential equations: if $\mathbf{x}(t) \in \mathbb{R}^n$ satisfies an equation of the form

$$\ddot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$$

for some smooth family of linear maps $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$, then $\mathbf{x}(t)$ is uniquely determined by its initial position $\mathbf{x}(0)$ and velocity $\dot{\mathbf{x}}(0)$. Use this and Equation (2) to show that if λ satisfies (1) and there is a point $p \in M$ at which $\lambda_p = 0$ and $\nabla\lambda_p = 0$, then $\lambda \equiv 0$.

(f) The previous conclusion together with the linearity of the Killing equation imply a uniqueness statement for the Killing equation: in particular, if M is connected, there is an upper bound (in terms of $\dim M = n$) on the possible dimension of the space of Killing vector fields. What is this bound?

Caution: This is a uniqueness result but says nothing about existence—there are cases where the Killing equation has no nontrivial solutions. The trouble is that while the theory of ODEs guarantees local existence of 1-forms λ that satisfy Equation (2) along a geodesic γ, these need not generally extend to 1-forms on an open set that satisfy (1).

(g) Let us apply the uniqueness result to the case $M = \mathbb{R}^n$ with the standard Euclidean metric $\langle\ ,\ \rangle$ on $T_p\mathbb{R}^n \cong \mathbb{R}^n$. In this case there is a well known family of isometries called the *Euclidean group* $\mathrm{E}(n)$, which consists of all diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

for $\mathbf{A} \in \mathrm{O}(n)$ and $\mathbf{b} \in \mathbb{R}^n$. Differentiating any smooth 1-parameter family $\varphi_t \in \mathrm{E}(n)$ with $\varphi_0 = \mathrm{Id}$ gives a Killing vector field

$$X(\mathbf{x}) = \left.\frac{d}{dt}\varphi_t(\mathbf{x})\right|_{t=0}.$$

Show that *all* Killing vector fields on Euclidean n-space are of this form.