PROBLEM SET 13
SOLUTIONS TO PROBLEM 2

2. (a) Choose a smooth path $\gamma(s) \in M$ with $\dot\gamma(0) = Y \in T_pM$, and extend it using the flow φ_X^t of X to a smooth map

$$\alpha(s,t) = \varphi_X^t(\gamma(s))$$

for real parameters s and t near 0. This satisfies $\alpha(s,0) = \gamma(s)$, $\partial_t\alpha(s,t) = X(\alpha(s,t))$ and $\partial_s\alpha(0,t) = T\varphi_X^t(Y)$. Then

$$(\mathcal{L}_X\lambda)(Y) = \frac{d}{dt}\lambda(T\varphi_X^t(Y))\bigg|_{t=0} = \frac{d}{dt}\lambda(\partial_s\alpha(0,t))\bigg|_{t=0}.$$

Regarding $\partial_s\alpha(0,t)$ as a vector field along the path $t \mapsto \alpha(0,t) = \varphi_X^t(p)$, we can apply a Leibnitz rule for the covariant derivative and transform the latter expression into

$$\begin{aligned}(\nabla_t\lambda)(\partial_s\alpha(0,t))|_{t=0} + \lambda(\nabla_t\partial_s\alpha(0,t)|_{t=0} &= (\nabla_X\lambda)(Y) + \lambda(\nabla_t\partial_s\alpha(0,0))\\ &= (\nabla_X\lambda)(Y) + \lambda(\nabla_s\partial_t\alpha(0,0)) = (\nabla_X\lambda)(Y) + \lambda(\nabla_Y X).\end{aligned}$$

Note: There is another way to prove the desired identity in this part, though it seems less suitable for the generalization required in part (b). But indeed, since λ is a differential 1-form, one can use Cartan's formula $\mathcal{L}_X\lambda = d\iota_X\lambda + \iota_X d\lambda = d(\lambda(X)) + d\lambda(X,\cdot)$, giving

$$\begin{aligned}(\mathcal{L}_X\lambda)(Y) &= \mathcal{L}_Y(\lambda(X)) + d\lambda(X,Y) = \mathcal{L}_Y(\lambda(X)) + \mathcal{L}_X(\lambda(Y)) - \mathcal{L}_Y(\lambda(X)) - \lambda([X,Y])\\ &= \mathcal{L}_X(\lambda(Y)) - \lambda(\nabla_X Y - \nabla_Y X)\\ &= (\nabla_X\lambda)(Y) + \lambda(\nabla_X Y) - \lambda(\nabla_X Y) + \lambda(\nabla_Y X)\\ &= (\nabla_X\lambda)(Y) + \lambda(\nabla_Y X).\end{aligned}$$

Here we used the formula $d\lambda(X,Y) = \mathcal{L}_X(\lambda(Y)) - \mathcal{L}_Y(\lambda(X)) - \lambda([X,Y])$ in the first line and then used the symmetry of the connection to replace the Lie bracket by $\nabla_X Y - \nabla_Y X$.

(b) For a more general covariant tensor field $T \in \Gamma(T_k^0 M)$, we apply a similar argument as in part (a), constructing for each $Y_1, \ldots, Y_k \in T_pM$ a map $\alpha_j(s,t)$ such that $\partial_s\alpha_j(0,t) = T\varphi_X^t(Y_j)$, which defines a vector field along the path $t \mapsto \varphi_X^t(p)$. The argument is then the same as in part (a), except the Leibnitz rule gives a separate term for each Y_j.

(c) Choose any vector field Z and take the Lie derivative of $g(X,Z)$ in the direction of Y: using the various Leibnitz rules,

$$\begin{aligned}\mathcal{L}_Y[g(X,Z)] &= g(\nabla_Y X, Z) + g(X, \nabla_Y Z) = (\nabla_Y X)^\flat(Z) + X^\flat(\nabla_Y Z)\\ &= \mathcal{L}_Y\left[X^\flat(Z)\right] = (\nabla_Y(X^\flat))(Z) + X^\flat(\nabla_Y Z),\end{aligned}$$

thus $(\nabla_Y X)^\flat = \nabla_Y(X^\flat)$.

In light of this, we shall from now on drop unnecessary parentheses. The tensor field $\nabla X^\flat$ is antisymmetric if

$$0 = \nabla X^\flat(Y,Z) + \nabla X^\flat(Z,Y) = \nabla_Y X^\flat(Z) + \nabla_Z X^\flat(Y) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),$$

which is the Killing equation.

(d) Note that if we are trying to show $(\nabla_s^2\lambda)(Y) = \lambda(R(\dot\gamma, Y)\dot\gamma)$, both sides of this expression are C^∞-linear with respect to Y, so it suffices to establish that for every s_0 in the domain of the geodesic γ and every vector $Y_0 \in T_{\gamma(s_0)}M$, there exists an extension of Y_0 to a smooth vector field Y along γ satisfying $Y(s_0) = Y_0$ and $(\nabla_s^2 \lambda)(Y) = \lambda(R(\dot\gamma, Y)\dot\gamma)$. In particular, we are free to choose $Y(s) \in T_{\gamma(s)}M$ to be a *parallel* vector field along γ, so that

$$\partial_s[\lambda(Y)] = (\nabla_s \lambda)(Y) + \lambda(\nabla_s Y) = (\nabla_s \lambda)(Y)$$

and similarly $\partial_s^2[\lambda(Y)] = (\nabla_s^2 \lambda)(Y)$. Our aim will thus be to show that if Y is parallel and $\nabla\lambda$ is antisymmetric,

$$\partial_s^2 \left(\lambda(Y)\right) = \lambda(R(\dot\gamma, Y)\dot\gamma).$$

Assume $\alpha(s,t)$ is a smooth map defined for real parameters s and t near 0 such that $\alpha(s,0) = \gamma(s)$ and $\partial_t \alpha(s,0) = Y(s)$; such a map can easily be constructed, e.g. via the exponential map

$$\alpha(s,t) = \exp_{\gamma(s)}(tY(s)).$$

Now

$$\begin{aligned}\partial_s \left(\lambda(Y)\right) &= (\nabla_s \lambda)(\partial_t \alpha(s,0)) = \nabla\lambda(\partial_s \alpha(s,0), \partial_t \alpha(s,0)) = -\nabla\lambda(\partial_t \alpha(s,0), \partial_s \alpha(s,0)) \\ &= -(\nabla_t \lambda)(\partial_s \alpha(s,0)),\end{aligned}$$

while

$$\partial_t \left(\lambda(\partial_s \alpha(s,t))\right)\big|_{t=0} = (\nabla_t \lambda)(\partial_s \alpha(s,0)) + \lambda(\nabla_t \partial_s \alpha(s,0)).$$

Note that $\nabla_t \partial_s \alpha(s,0) = \nabla_s \partial_t \alpha(s,0) = \nabla_s Y(s) = 0$, thus combining these two expressions gives

$$\partial_s \left(\lambda(Y)\right) = -\left.\partial_t \left(\lambda(\partial_s \alpha(s,t))\right)\right|_{t=0}.$$

Differentiating this with respect to s and exchanging ∂_s with ∂_t,

$$\partial_s^2 \left(\lambda(Y)\right) = -\left.\partial_t \partial_s \left(\lambda(\partial_s \alpha(s,t))\right)\right|_{t=0}.$$

The antisymmetry of $\nabla\lambda$ also implies $(\nabla_s \lambda)(\partial_s \alpha(s,t)) = \nabla\lambda(\partial_s \alpha(s,t), \partial_s \alpha(s,t)) = 0$, thus

$$\partial_s \left(\lambda(\partial_s \alpha(s,t))\right) = (\nabla_s \lambda)(\partial_s \alpha(s,t)) + \lambda(\nabla_s \partial_s \alpha(s,t)) = \lambda(\nabla_s \partial_s \alpha(s,t)),$$

and inserting this into the previous expression, together with the fact that $\gamma(s) = \alpha(s,0)$ is a geodesic,

$$\begin{aligned}\partial_s^2 \left(\lambda(Y)\right) &= -\left.\partial_t \left(\lambda(\nabla_s \partial_s \alpha(s,t))\right)\right|_{t=0} = -(\nabla_t \lambda)(\nabla_s \partial_s \alpha(s,0)) - \lambda(\nabla_t \nabla_s \partial_s \alpha(s,0)) \\ &= -\lambda(\nabla_t \nabla_s \partial_s \alpha(s,0)).\end{aligned}$$

Finally, we use the Riemann tensor to interchange covariant partials:

$$\nabla_s \nabla_t \partial_s \alpha(s,0) - \nabla_t \nabla_s \partial_s \alpha(s,0) = R(\partial_s \alpha(s,0), \partial_t \alpha(s,0))\partial_s \alpha(s,0) = R(\dot\gamma(s), Y(s))\dot\gamma(s),$$

and thus $-\nabla_t \nabla_s \partial_s \alpha(s,0) = R(\dot\gamma(s), Y(s))\dot\gamma(s) - \nabla_s \nabla_t \partial_s \alpha(s,0)$. This last term is also

$$-\nabla_s \nabla_t \partial_s \alpha(s,0) = -\nabla_s \nabla_s \partial_t \alpha(s,0) = -\nabla_s^2 Y(s) = 0$$

since $Y(s)$ is parallel. We conclude

$$\partial_s^2 \left(\lambda(Y)\right) = \lambda(R(\dot\gamma(s), Y(s))\dot\gamma(s))$$

as claimed.

(e) Assume $\nabla\lambda$ is antisymmetric and there is a point $p \in M$ at which $\lambda_p = 0$ and $\nabla\lambda_p = 0$. Then along any geodesic $\gamma(s)$ through $\gamma(0) = p$, λ satisfies the second order linear differential equation of part (d). Choosing a coordinate chart $(x^1, \ldots, x^n)$ near p, this equation can be expressed via the components λ_i in the form

$$\frac{d^2}{ds^2}\lambda_i(\gamma(s)) = A_i^j(s)\lambda_j(\gamma(s)),$$

where $A_i^j(\gamma(s))$ is a set of smooth functions determined by the components of the Riemann tensor along $\gamma(s)$. Since $\lambda_i(\gamma(0))$ and $\frac{d}{ds}\lambda_i(\gamma(s))\big|_{s=0}$ both vanish, the unique solution to this differential equation is $\lambda_i(\gamma(s)) \equiv 0$. This shows that λ vanishes in an open neighborhood of p, as one can use geodesics through p to hit every point in such a neighborhood. If M is connected, it follows that $\lambda \equiv 0$.

The statement of the problem should really have assumed that M is connected, but you probably figured that out.

(f) By part (e), there is a unique solution $\lambda \equiv 0$ such that both λ and $\nabla\lambda$ vanish at any given point $p \in M$. Since the Killing equation is linear, it follows that any two solutions λ_1 and λ_2 that match up to first order at p are identical. Recalling that $\nabla\lambda_p \in \Lambda^2 T_p^* M$ due to the Killing equation, the space of Killing vector fields is therefore no larger than the space $T_p^*M \oplus \Lambda^2 T_p^* M$, which has dimension

$$n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

(g) The Euclidean group has dimension

$$\dim \mathrm{E}(n) = \dim \mathrm{O}(n) + \dim \mathbb{R}^n = \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2},$$

and this gives rise to a $\frac{n(n+1)}{2}$-dimensional space of Killing vector fields. By part (f), this is the largest possible dimension of such a space, and is therefore all of them.

As a matter of interest: one can use arguments similar to those of part (d) to show that the Killing equation is equivalent to a certain n-dimensional distribution on the vector bundle $T^*M \oplus \Lambda^2 T^*M \to M$, and in fact defines a linear connection on this bundle. Then a Killing vector field is equivalent to a flat section of $T^*M \oplus \Lambda^2 T^*M$ with respect to this special connection. Generically, there will be no such flat sections, and thus no Killing vector fields, because the special connection may have nontrivial curvature. On the other hand, a space of Killing vector fields with the maximal allowed dimension will exist if the special connection is flat, which is true only if (M, g) is particularly symmetric, e.g. has constant curvature. These considerations are fundamental in cosmology, which starts from the assumption that the global structure of the universe is as symmetric as possible.