PROBLEM SET 14

Suggested reading

Lecture notes (on the website): the rest of Chapter 6

Problems

1. Assume (Σ, h) is a Riemannian 2-manifold whose Gaussian curvature is everywhere nonpositive. The statements in parts (a) and (b) below are each slightly weakened versions of standard results about geodesics in fixed homotopy classes on manifolds with nonpositive sectional curvature; both can be proved using the Gauss-Bonnet formula.

   (a) Show that (Σ, h) does not admit any *periodic* geodesic (i.e. a geodesic $\gamma : \mathbb{R} \to \Sigma$ satisfying $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$ and some fixed $T > 0$) whose image bounds an embedded disk.

   (b) Given a pair of distinct points $p, q \in \Sigma$ and a pair of geodesic segments $\gamma_0, \gamma_1 : [0,1] \to \Sigma$ with $\gamma_0(0) = \gamma_1(0) = p$ and $\gamma_0(1) = \gamma_1(1) = q$, show that there does not exist any smooth family of paths $\{\gamma_s : [0,1] \to \Sigma\}_{s \in [0,1]}$ from p to q, matching the given geodesics for $s = 0, 1$, such that the map $[0,1] \times (0,1) \to \Sigma : (s,t) \mapsto \gamma_s(t)$ is an embedding.

   (c) Find an example of a periodic geodesic on a surface with nonpositive Gaussian curvature. *(Note that by part (a), it had better not form the boundary of an embedded disk.)*

   (d) Show that the phenomenon ruled out by part (b) can actually happen on S^2.

2. Consider a smooth vector bundle $\pi : E \to M$ with a connection ∇, and suppose $f : N \to M$ is a smooth map between manifolds. Recall that the *pullback connection* on the bundle $f^*E \to N$ is defined as the unique connection for which the parallel transport

$$P^t_\gamma : (f^*E)_{\gamma(0)} = E_{f(\gamma(0))} \to E_{f(\gamma(t))} = (f^*E)_{\gamma(t)}$$

   along a path $\gamma(t) \in N$ is given simply by the parallel transport on E along $f \circ \gamma(t) \in M$ defined via ∇. Recall also that any local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ on E over an open subset $\mathcal{U}_\alpha \subset M$ gives rise to a local trivialization of f^*E over $\mathcal{V}_\alpha := f^{-1}(\mathcal{U}_\alpha)$, namely

$$\Psi_\alpha : (f^*E)|_{\mathcal{V}_\alpha} \to \mathcal{V}_\alpha \times \mathbb{F}^m$$

   such that if $x \in \mathcal{V}_\alpha$, $v \in (f^*E)_x = E_{f(x)}$ and $\Phi_\alpha(v) = (f(x), v_\alpha)$, then $\Psi_\alpha(v) = (x, v_\alpha)$. Note that if the sets $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ form an open covering of M, then $\{\mathcal{V}_\alpha\}_{\alpha \in I}$ form an open covering of N since f is continuous.

   (a) Show that if $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$ denotes the local connection 1-form associated to ∇ and the trivialization Φ_α, then the connection 1-form for the pullback connection on f^*E with respect to the local trivialization Ψ_α is $f^*A_\alpha \in \Omega^1(\mathcal{V}_\alpha, \mathfrak{g})$.

   (b) Assuming E is a complex line bundle, show that its first Chern class $c_1(E) \in H^2_{\mathrm{dR}}(M)$ has the following naturality property:

$$c_1(f^*E) = f^*c_1(E) \in H^2_{\mathrm{dR}}(N).$$

   (c) Show that if E and F are two complex line bundles over the same manifold M admitting a complex bundle isomorphism $E \to F$, then $c_1(E) = c_1(F)$.

3. For this problem, we will assume $E \to \Sigma$ is a complex line bundle over a closed oriented surface, and abuse notation by abbreviating its first Chern number as

$$c_1(E) := \int_\Sigma c_1(E) \in \mathbb{R},$$

where the integral on the right hand side should be interpreted to mean $\int_\Sigma \omega$ for any choice of closed 2-form ω with $[\omega] = c_1(E) \in H^2_{\mathrm{dR}}(\Sigma)$. As shown in Thursday's lecture, the number $c_1(E)$ is always an integer, and it equals the signed and weighted count of zeroes for any section $s \in \Gamma(E)$ that is nonzero outside of finitely many points, i.e.

$$c_1(E) = \sum_{z \in s^{-1}(0)} \mathrm{ind}(s; z),$$

where for each $z \in s^{-1}(0)$, $\mathrm{ind}(s; z) \in \mathbb{Z}$ is the winding number along the (positively oriented!) boundary of a small disk containing z, for the complex-valued function representing s with respect to a local trivialization on that disk. Using this fact, prove:

(a) $c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2)$ for any pair of complex line bundles $E_1, E_2 \to \Sigma$.

(b) The complex dual bundle $E^* \to \Sigma$ satisfies $c_1(E^*) = -c_1(E)$.
Hint: See Problem Set 9 #3(c).

(c) Let Σ_g denote the closed oriented surface of genus g, and define $E \to \Sigma_g$ to be the complex line bundle obtained by endowing $T\Sigma_g \to \Sigma_g$ with the complex structure $j : T\Sigma_g \to T\Sigma_g$ that rotates vectors by 90 degrees (in the positive direction with respect to the orientation). Show that E is isomorphic to its own complex dual bundle if and only if $g = 1$.