PROBLEM SET 4

Suggested reading

As usual, chapter and section indications in Lee refer to the 2003 edition and may differ in the 2013 edition.

- Baum: §2.7
- Friedrich and Agricola: Chapter 1
- Lee: Chapter 11 ("Symmetric Tensors" and the first few pages of "Riemannian Metrics"—we'll come back to that section later), Chapter 12 (up to "Differential Forms on Manifolds"), Chapter 2 ("Partitions of Unity")

Note: Baum and Lee both discuss partitions of unity using the topological notion of *second countability*, which implies that a manifold is *paracompact*. In lecture we have not discussed second countability but have instead required all manifolds to be metrizable, and this is one of the very few places in the course where that requirement becomes important, simply because all metric spaces are paracompact. For a thorough treatment of partitions of unity that is closer to the sketch we will see in lecture, see the end of Chapter 2 in Volume 1 of Spivak's *Comprehensive Introduction to Differential Geometry*. Appendix A in Spivak also provides an entertaining example of an object that would be a 1-dimensional manifold if we did not require them to be metrizable, the so-called "long line". It does not admit partitions of unity, and I guarantee that you will not be able to visualize it.

Problems

Note that in the following problems and throughout the rest of the course, we sometimes use the Einstein summation convention without mentioning it explicitly.

1. Recall that in lecture we used the concept of C^∞*-linearity* to prove that for any dual vector field $\lambda \in \Gamma(T_1^0 M)$ on a smooth manifold M, the bilinear map $T : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to C^\infty(M)$ defined by

$$T(X, Y) = L_X(\lambda(Y)) - L_Y(\lambda(X)) - \lambda([X, Y])$$

defines a tensor. Let's be clear on the meaning of this statement: in the above expression, X and Y are vector fields, thus $\lambda(X)$ is the smooth real-valued function $p \mapsto \lambda(X(p))$, and its Lie derivative with respect to Y is another smooth real valued function on M, as is $\lambda([X, Y])$. The entire expression thus defines a real-valued function, and when we say it *defines a tensor*, we mean that the value of this function at each point $p \in M$ depends only on $X(p)$ and $Y(p)$, not on any extra information about X and Y as vector fields (e.g. their derivatives). Clearly any bilinear map with these properties satisfies

$$T(fX, Y) = f \cdot T(X, Y) = T(X, fY)$$

for all C^∞ functions $f : M \to \mathbb{R}$, and in lecture we proved a converse to this statement: any multilinear map

$$A : \underbrace{\mathrm{Vec}(M) \times \ldots \times \mathrm{Vec}(M)}_{k} \to C^\infty(M)$$

that is C^∞-linear in this sense with respect to each variable defines a tensor field $A \in \Gamma(T_k^0 M)$.

   (a) Choosing local coordinates $(x^1, \ldots, x^n)$ near some point $p \in M$ and writing $\lambda = \lambda_i \, dx^i$ and $T = T_{ij} \, dx^i \otimes dx^j$, show that the component functions λ_i and T_{ij} are related by

$$T_{ij} = \partial_i \lambda_j - \partial_j \lambda_i.$$

   You may find it helpful to recall that brackets of coordinate vector fields always vanish, that is, $[\partial_i, \partial_j] \equiv 0$.

(b) Show that if $\lambda \in \Gamma(T_1^0 M)$ is not identically zero, then there exists no tensor field $S \in \Gamma(T_2^0 M)$ that is related to λ by

$$S_{ij} = \partial_i \lambda_j$$

for all choices of local coordinates.

(c) Recall from Problem Set 3 #2 that for any finite-dimensional vector space V, there is a canonical isomorphism between the space of multilinear maps $\underbrace{V \times \ldots \times V}_{k} \times V^* \to \mathbb{R}$ and the space of multilinear maps $\underbrace{V \times \ldots \times V}_{k} \to V$, hence tensor fields $A \in \Gamma(T_k^1 M)$ of type $(1, k)$ can naturally be viewed as associating to each $p \in M$ a multilinear map

$$A_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \to T_pM.$$

In particular, this gives rise to a multilinear map

$$A : \underbrace{\mathrm{Vec}(M) \times \ldots \times \mathrm{Vec}(M)}_{k} \to \mathrm{Vec}(M) \tag{1}$$

which is C^∞-linear in each variable. Prove the converse: any multilinear map as in (1) that is C^∞-linear in each variable defines a tensor field $A \in \Gamma(T_k^1 M)$.

2. In this problem, denote the entries of an n-by-n matrix $\mathbf{A}$ by $\mathbf{A}^i{}_j$. Thus multiplication of two n-by-n matrices can be expressed using the summation convention as

$$(\mathbf{AB})^i{}_j = \mathbf{A}^i{}_k \mathbf{B}^k{}_j.$$

The *trace* of a matrix is the scalar $\mathrm{tr}(\mathbf{A})$ obtained by summing the diagonal entries: using the summation convention,

$$\mathrm{tr}(\mathbf{A}) = A^i{}_i.$$

(a) Show that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ for any pair of n-by-n matrices.

(b) Use the above to show that $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{BAB}^{-1})$.

(c) If $A \in \Gamma(T_1^1 M)$ is a tensor field of type $(1, 1)$, we define the *contraction* of A to be the smooth real valued function $\mathrm{tr}\, A : M \to \mathbb{R}$ which equals

$$\mathrm{tr}\, A = A^i{}_i,$$

where $A^i{}_j$ are the components of A in any local coordinate system. In other words, to compute $\mathrm{tr}\, A(p)$ for $p \in M$, we pick a coordinate chart on a neighborhood $\mathcal{U}$ of p, write down the corresponding component functions $A^i{}_j : \mathcal{U} \to \mathbb{R}$ and compute the above expression at p. Explain why the result is independent of the choice of chart. (See the discussion at the end of Problem Set 3 #3(e).)

For mixed tensors of higher rank there are more general contractions that can be defined: e.g. from a tensor field of type $(4, 2)$ with components $T^{ijk\ell}{}_{pr}$, one can define one of type $(3, 1)$ whose components are

$$S^{ijk}{}_p := T^{ijk\ell}{}_{p\ell}.$$

By a slight extension of the argument for tensors of type $(1, 1)$, such operations give well defined homomorphisms

$$T_\ell^k M \to T_{\ell-1}^{k-1} M.$$

3. Define a 1-form on $\mathbb{R}^2 \setminus \{0\}$ in the standard (x, y)-coordinates by

$$\lambda = \frac{1}{x^2 + y^2}(x\,dy - y\,dx),$$

and let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ denote the natural embedding of the unit circle in $\mathbb{R}^2$.

(a) Using a finite covering by charts and a subordinate partition of unity, compute

$$\int_{S^1} \iota^* \lambda = 2\pi.$$

(b) Can you give the result of part (a) a nice interpretation in terms of polar coordinates?

(c) Show that for any smooth function $f : S^1 \to \mathbb{R}$, the 1-form $df \in \Omega^1(S^1)$ satisfies

$$\int_{S^1} df = 0.$$