PROBLEM SET 7

Suggested reading

Lecture notes (on the website): Appendix B, *Lie groups and Lie algebras*

Problems

1. The goal of this problem is to prove Cartan's formula for the Lie derivative of a differential form,[1]

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega. \tag{1}$$

Recall that for a k-form $\omega \in \Omega^k(M)$ on a smooth n-manifold M, $\mathcal{L}_X \omega \in \Omega^k(M)$ is defined as $\partial_t \varphi_t^* \omega|_{t=0}$ where $\varphi_t : M \to M$ denotes the time t flow of the vector field $X \in \mathrm{Vec}(M)$.

(a) Use the definition of the Lie derivative to prove the relation

$$d(\mathcal{L}_X \omega) = \mathcal{L}_X(d\omega)$$

and the Leibniz rule[2]

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta. \tag{2}$$

Observe that this formula determines the action of $\mathcal{L}_X$ on arbitrary differential forms if we know how it acts on 0-forms (i.e. smooth functions) and exact 1-forms (i.e. differentials of smooth functions).

(b) Show that (1) holds for all 0-forms and exact 1-forms.

(c) Fix an n-dimensional vector space V with basis $v_1, \dots, v_n$ and dual basis $\lambda^1, \dots, \lambda^n$. For $v \in V$, define $\iota_v : \Lambda^k V^* \to \Lambda^{k-1} V^*$ by $\iota_v \alpha := \alpha(v, \cdots)$. Show that for any $1 \le i_1 < \ldots < i_k \le n$ and $j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$, if $\alpha := \lambda^{i_1} \wedge \ldots \wedge \lambda^{i_k}$, then

$$\iota_{v_j} \alpha = 0 \quad \text{and} \quad \iota_{v_j}(\lambda^j \wedge \alpha) = \alpha.$$

Use this to deduce that for any fixed $v \in V$, ι_v satisfies the graded Leibniz rule

$$\iota_v(\alpha \wedge \beta) = \iota_v \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \iota_v \beta.$$

Hint: Show first that the formula holds whenever v is one of the basis vectors and α and β are both wedge products of dual basis vectors. Then appeal to the multilinearity of the map $V \times \Lambda^k V^* \to \Lambda^{k-1} V^* : (v, \alpha) \mapsto \iota_v \alpha$ to deduce the general case.

(d) For any fixed $X \in \mathrm{Vec}(M)$, use the graded Leibniz rules satisfied by d and ι_X to show that the operator $(d \circ \iota_X + \iota_X \circ d) : \Omega^k(M) \to \Omega^k(M)$ also satisfies the Leibniz rule (2). Deduce that this operator matches $\mathcal{L}_X$.

2. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. In this problem we will show that various commonly encountered groups of matrices are Lie groups, because they are both subgroups and smooth submanifolds of the general linear group $\mathrm{GL}(n, \mathbb{F})$. Recall that the latter is a Lie group because it is an open subset (and hence a submanifold) of the vector space $\mathbb{F}^{n \times n} \cong \mathbb{F}^{n^2}$ of n-by-n matrices with entries in $\mathbb{F}$, and the algebraic

[1] From now on, we use $\mathcal{L}_X$ to denote the Lie derivative with respect to a vector field X; I've denoted it by L_X in previous problem sets and by something more like $\mathscr{L}_X$ in lectures. I'm changing the font here so that it doesn't get confused with the left-multiplication diffeomorphisms $L_g : G \to G$ defined on a Lie group G.

[2] Notice that in contrast to the exterior derivative and the interior product treated in part (c), the Leibniz rule satisfied by $\mathcal{L}_X$ does not include any annoying signs. This is consistent with our usual mnemonic if we think of $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ as an object of degree zero (hence even), while $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is an object of degree one (hence odd), so that exchanging the order of d with a form of odd degree causes a sign change, but no such thing happens with $\mathcal{L}_X$. The mnemonic applies similarly to $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by thinking of the latter as an object of degree -1, hence odd.

operations $\mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n} : (\mathbf{A}, \mathbf{B}) \mapsto \mathbf{AB}$ and $\mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(n, \mathbb{F}) : \mathbf{A} \mapsto \mathbf{A}^{-1}$ are smooth maps. We showed already in Problem Set 2 #1 that $\mathrm{O}(n) := \{\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) \mid \mathbf{A}^T\mathbf{A} = \mathbb{1}\}$ is a smooth submanifold of $\mathrm{GL}(n, \mathbb{R})$ with dimension $n(n-1)/2$ and $\mathfrak{o}(n) := T_{\mathbb{1}} \mathrm{O}(n) = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A} + \mathbf{A}^T = 0\}$.

We continue now with the special linear group $\mathrm{SL}(n, \mathbb{F}) := \{\mathbf{A} \in \mathbb{F}^{n\times n} \mid \det(\mathbf{A}) = 1\}$, which is a level set of the map

$$\det : \mathbb{F}^{n\times n} \to \mathbb{F}.$$

Note that the latter is a polynomial function of the matrix entries, so it is clearly a smooth map; if we can show that its derivative $d(\det)(\mathbf{A}) : \mathbb{F}^{n\times n} \to \mathbb{F}$ is a surjective whenever $\det(\mathbf{A}) = 1$, then the implicit function theorem implies that $\mathrm{SL}(n, \mathbb{F})$ is a submanifold.

(a) If $\mathbf{A}(t) \in \mathbb{F}^{n\times n}$ is a smooth path of matrices with $\mathbf{A}(0) = \mathbb{1}$ and its time derivative is denoted by $\dot{\mathbf{A}}(t)$, show that

$$\left.\frac{d}{dt} \det(\mathbf{A}(t))\right|_{t=0} = \mathrm{tr}(\dot{\mathbf{A}}(0)). \tag{3}$$

Hint: Think of $\mathbf{A}(t)$ *as an* n*-tuple of column vectors*

$$\mathbf{A}(t) = \begin{pmatrix} \mathbf{v}_1(t) & \cdots & \mathbf{v}_n(t) \end{pmatrix}$$

with $\mathbf{v}_j(0) = \mathbf{e}_j$*, the standard basis vector. Then* $\det(\mathbf{A}(t))$ *is the evaluation of an alternating* n*-form on these vectors, which can be written using components. Write it this way and use the product rule.*

(b) Show that if $\mathbf{A} \in \mathrm{GL}(n, \mathbb{F})$ then the derivative of $\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ at $\mathbf{A}$ is

$$d(\det)(\mathbf{A})\mathbf{H} = \det(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{A}^{-1}\mathbf{H}).$$

(c) Show that the aforementioned derivative is surjective, implying that $\det^{-1}(1) \subset \mathbb{F}^{n\times n}$ is a smooth submanifold of dimension $n^2 - 1$ if $\mathbb{F} = \mathbb{R}$, or $2n^2 - 2$ if $\mathbb{F} = \mathbb{C}$.

(d) Show that $\mathfrak{sl}(n, \mathbb{F}) := T_{\mathbb{1}} \mathrm{SL}(n, \mathbb{F}) = \{\mathbf{A} \in \mathbb{F}^{n\times n} \mid \mathrm{tr}(\mathbf{A}) = 0\}$.

(e) Adapt the argument of Problem Set 2 #1 to show that the unitary group $\mathrm{U}(n) := \{\mathbf{A} \in \mathrm{GL}(n, \mathbb{C}) \mid \mathbf{A}^\dagger\mathbf{A} = \mathbb{1}\}$ is a smooth n^2-dimensional submanifold of $\mathrm{GL}(n, \mathbb{C})$ with $\mathfrak{u}(n) := T_{\mathbb{1}} \mathrm{U}(n) = \{\mathbf{A} \in \mathbb{C}^{n\times n} \mid \mathbf{A} + \mathbf{A}^\dagger = 0\}$.

(f) Show that the special unitary group $\mathrm{SU}(n) := \{\mathbf{A} \in \mathrm{U}(n) \mid \det(\mathbf{A}) = 1\}$ is a smooth $(n^2 - 1)$-dimensional submanifold of $\mathrm{U}(n)$.
Hint: What is the image of the map $\det : \mathrm{U}(n) \to \mathbb{C}$*?*

3. You may by wondering why, when we talk about a general Lie group G with identity element $e \in G$, we tend to talk about left-invariant vector fields on G without ever mentioning *right-invariant* vector fields. The upshot of this problem will be that it doesn't really matter: both notions are equally good for the main things we want them for, namely defining the exponential map and the Lie algebra structure of $\mathfrak{g} := T_eG$.

Let's denote $R_g : G \to G : h \mapsto hg$ for $g \in G$, and call $X \in \mathrm{Vec}(G)$ right-invariant if it satisfies

$$X(R_g(h)) = (R_g)_* X(h)$$

for all $g, h \in G$.

(a) Show that for every $X \in \mathfrak{g}$, there is a unique right-invariant vector field $X^R \in \mathrm{Vec}(G)$ satisfying $X^R(e) = X$.

(b) Show that if $X, Y \in \mathrm{Vec}(G)$ are both right-invariant, then so is $[X, Y] \in \mathrm{Vec}(G)$.

(c) Given $X \in \mathfrak{g}$, let X^L denote the unique left-invariant vector field with $X^L(e) = X$. Show that $X^L = X^R$ everywhere along the image of the curve $t \mapsto \exp(tX)$, and the latter is an integral curve (i.e. a flow line) of both.

(d) Show that for any $X, Y \in \mathfrak{g}$ and $f \in C^\infty(G)$,

$$\mathcal{L}_{X^R}\mathcal{L}_{Y^R} f(e) = \partial_s \partial_t f(\exp(tY)\exp(sX))\big|_{s=t=0}.$$

(e) Comparing the formula in part (d) with the corresponding formula involving left-invariant vector fields X^L and Y^L, show that for all $X, Y \in \mathfrak{g}$,

$$[X^L, Y^L](e) = -[X^R, Y^R](e).$$

The message of this result is the following: whether we choose to define $[\ ,\ ]$ *on* $\mathfrak{g}$ *using left-invariant or right-invariant vector fields, this choice only makes a difference of a sign. In actuality neither choice is better than the other, just as the commutator bracket for matrices could just as well be defined by* $[\mathbf{A}, \mathbf{B}] = \mathbf{BA} - \mathbf{AB}$ *instead of* $\mathbf{AB} - \mathbf{BA}$*. But the latter choice is the established convention, so in order to stay consistent with it, we use* left*-invariant vector fields to define* $[\ ,\ ]$ *on* $\mathfrak{g}$.

4. (a) Use Leibniz rules as in Problem 1 to show that for all $X, Y \in \mathrm{Vec}(M)$ and $\omega \in \Omega^k(M)$,

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega.$$

(b) For any closed manifold M and differential form $\omega \in \Omega^k(M)$, one can define a subgroup of the topological group $\mathrm{Diff}(M)$ of all diffeomorphisms $M \to M$ by

$$\mathrm{Diff}(M, \omega) := \left\{ \varphi \in \mathrm{Diff}(M) \;\middle|\; \varphi^*\omega = \omega \right\}.$$

For example, $\mathrm{Diff}(M, \omega) = \mathrm{Diff}(M)$ if ω is identically zero. Show that if ω is a volume form on M and we define volumes of domains $\mathcal{U} \subset M$ by $\mathrm{Vol}(\mathcal{U}) := \int_{\mathcal{U}} \omega$, then $\mathrm{Diff}(M, \omega)$ is the group of orientation-preserving diffeomorphisms of M that also preserve volumes, i.e. that satisfy

$$\mathrm{Vol}(\mathcal{U}) = \mathrm{Vol}(\varphi(\mathcal{U}))$$

for all domains $\mathcal{U} \subset M$.

(c) The example $\omega = 0$ shows that $\mathrm{Diff}(M, \omega)$ cannot be expected to be finite-dimensional and will thus generally not be a Lie group. But formally, we can pretend it is one and define its "Lie algebra" as the vector space

$$\mathfrak{diff}(M, \omega) := \left\{ X \in \mathrm{Vec}(M) \;\middle|\; X = \partial_t \varphi_t|_{t=0} \text{ for some } \{\varphi_t \in \mathrm{Diff}(M, \omega)\}_{t \in (-\epsilon, \epsilon)} \text{ with } \varphi_0 = \mathrm{Id} \right\},$$

where the family of diffeomorphisms $\varphi_t : M \to M$ is assumed to depend smoothly on the parameter t. Show that $\mathfrak{diff}(M, \omega)$ contains the linear subspace $\{X \in \mathrm{Vec}(M) \mid \mathcal{L}_X \omega = 0\}$.[3]

(d) Show that if $X, Y \in \mathrm{Vec}(M)$ satisfy $\mathcal{L}_X \omega = \mathcal{L}_Y \omega = 0$, then $\mathcal{L}_{[X,Y]}\omega = 0$ as well. In other words, the subspace in part (c) is a *Lie subalgebra* of the Lie algebra of smooth vector fields $(\mathrm{Vec}(M), [\ ,\ ])$.

(e) A 2-form $\omega \in \Omega^2(M)$ is called *symplectic* if it satisfies $d\omega = 0$ and is *nondegenerate*, meaning that for every $p \in M$ and nonzero vector $X \in T_pM$, the linear map $\omega(X, \cdot) : T_pM \to \mathbb{R}$ is nonzero. Show that if ω is a symplectic form, then the space $\{X \in \mathrm{Vec}(M) \mid \mathcal{L}_X \omega = 0\}$ is infinite dimensional.

Hint: In lecture last Thursday we saw an example of a symplectic form on $\mathbb{R}^{2n}$ *and showed that for every smooth function* $H : \mathbb{R}^{2n} \to \mathbb{R}$*, the corresponding Hamiltonian flow preserves the symplectic form. Can you generalize this discussion to* M*?*

Second hint: The nondegeneracy condition implies that ω *defines an isomorphism* $\mathrm{Vec}(M) \to \Omega^1(M) : X \mapsto \iota_X \omega$*. Why?*

[3] Actually one can show that this subspace also contains $\mathfrak{diff}(M, \omega)$, but this requires one or two lemmas about the Lie derivative that we haven't proved, so never mind that for now.